

# **Unification of Fusion Theories and Rules, Filter Algorithms, and Target Tracking Methods for Applications to Air Force Problems**

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## **Abstract.**

In this paper we propose a unification method in order to be able to checking what fusion theory, what combination fusion rule, and what filtering method to use for data fusion, image fusion, and target tracking for each military application.

We thus try to solving air force problems using a combination of fusion theories, rules, and target tracking algorithms.

**Keywords:** UFR, UFT, filter algorithms, neutrosophic logic and set, fuzzy logic and set, data fusion, image fusion, non-linear sequences

## **Introduction.**

We focus on four topics for this research: unification of fusion theories and rules, unification of filter algorithms, data fusion and image fusion using neutrosophic logic, and non-linear sequences for target tracking.

### **1) *Unification of Fusion Theories and Rules (UFTR)***

Since no fusion theories neither rules fully work for all applications, we propose Unification of Fusion Theories and of combination of fusion rules in solving problems/applications. For each particular application, one selects the most appropriate model, rule(s), and algorithm of implementation.

We are working in the unification of the fusion theories and rules, which looks like a logical chart for a computer programmer.

The unification scenario should periodically be updated incorporating new discoveries from the fusion and engineering research.

Amongst fusion theories we remind the most known ones: Bayesian Theory, Dempster-Shafer Theory (DST), Transferable Belief Model (TBM), and Dezert-Smarandache Theory (DSmT).

### **2) *Unification of Filter Algorithms (UFA)***

The idea is to extend the NIS [Normalized Innovation Squared] procedure and take into consideration all filters to find out which works better.

### **3) *Data Fusion and Image Fusion Methods using Neutrosophic Logic***

Adjust the bba (basic belief assignment) within a neutrosophic framework, and then use the neutrosophic logical operators (such as “and”, “or”, negation) for combining information received from multiple sources.

Also, let's propose a method of using Neutrosophic Logic, which is a generalization of fuzzy logic and especially of intuitionistic fuzzy logic, to image segmentation and image thresholding. Improve the watershed algorithms using a neutrosophic approach {i.e. considering the objects as the set T (truth), the background as the set F (falsehood), and the edges as the set I (indeterminacy, since the cell's borders are vague)}; the method should be less sensitive to noise and perform better on non-uniform images since it uses the indeterminacy (I) from neutrosophic logic and set (this indeterminacy is not featured in fuzzy logic/set).

Use neutrosophic logic/set/probability/statistics as a new trend in image processing and compare the neutrosophic approach to other methods (such as: histogram-based, edge-based, region-based, model-based, watershed/topographic in MatLab or using Toboggan-Based).

Next step would be to use the neutrosophic approach to image registration and similarly compare the result with those obtained from other methods.

Interesting also is to use the neutrosophic approach to the control theory.

#### 4) *Non-Linear Sequences for Target Tracking*

Besides linear tracking methods would be to use non-linear sequences in non-linear tracking.

### 1) *Unification of Fusion Theories and Rules (UFTR)*

#### 1.1. Introduction.

Each theory works well for some applications, but not for all.

We extended the power set (from Dempster-Shafer Theory) and hyper-power set (from Dezert-Smarandache Theory) to a Boolean algebra called super-power set obtained by the closure of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers as complement the fuzzy or neutrosophic complement). All bba's (basic belief assignments) and rules are redefined on this Boolean algebra.

A similar generalization has been previously used by Guan-Bell (1993) for the Dempster-Shafer rule using propositions in sequential logic, herein we reconsider the Boolean algebra for all fusion rules and theories but using sets instead of propositions, because generally it is harder to work in sequential logic with summations and inclusions than in the set theory.

#### 1.2. Fusion Space.

For  $n \geq 2$  let  $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$  be the frame of discernment of the fusion problem/application under consideration. Then  $(\Theta, \cup, \cap, C)$ ,  $\Theta$  closed under these three operations: union, intersection, and complementation of sets respectively, forms a Boolean algebra. With respect to the partial ordering relation, the inclusion  $\subseteq$ , the minimum element is the empty set  $\phi$ , and the maximal element is the total

$$\text{ignorance } I = \bigcup_{i=1}^n \theta_i.$$

Similarly one can define:  $(\Theta, \cup, \cap, \setminus)$  for sets,  $\Theta$  closed with respect to each of these operations: union, intersection, and difference of sets respectively.

$(\Theta, \cup, \cap, C)$  and  $(\Theta, \cup, \cap, \setminus)$  generate the same super-power set  $S^\Theta$  closed under  $\cup, \cap, C$ , and  $\setminus$  because for any  $A, B \in S^\Theta$  one has  $CA = I \setminus A$  and reciprocally  $A \setminus B = A \cap CB$ .

If one considers propositions, then  $(\Theta, \vee, \wedge, \neg)$  forms a Lindenbaum algebra in sequential logic, which is isomorphic with the above  $(\Theta, \cup, \cap, C)$  Boolean algebra.

By choosing the frame of discernment  $\Theta$  closed under  $\cup$  only one gets DST, Yager's, TBM, DP theories. Then making  $\Theta$  closed under both  $\cup, \cap$  one gets DSm theory. While, extending  $\Theta$  for closure under  $\cup, \cap$ , and  $C$  one also includes the complement of set (or negation of proposition if working in sequential logic); in the case of non-exclusive elements in the frame of discernment one considers a fuzzy or neutrosophic complement. Therefore the super-power set  $(\Theta, \cup, \cap, C)$  includes all the previous fusion theories.

The power set  $2^\Theta$ , used in DST, Yager's, TBM, DP, which is the set of all subsets of  $\Theta$ , is also a Boolean algebra, closed under  $\cup, \cap$ , and  $C$ , but does not contain intersections of elements from  $\Theta$  since all elements are supposed exclusive.

The Dedekind distributive lattice  $D^\Theta$  (hyper-power set) used in DSmT, is closed under  $\cup, \cap$ , and if negations/complements arise they are directly introduced in the frame of discernment, say  $\Theta'$ , which is then closed under  $\cup, \cap$ . Unlike others, DSmT allows intersections, generalizing the previous theories.

The Unifying Theory contains intersections and complements as well.

In a particular case, for a frame of discernment with two elements  $\Theta = \{A, B\}$ , we remark that:

a) in classical probability (Bayesian) theory:

$$P(A) + P(B) = 1, \text{ where } P(X) \text{ means probability of } X; \quad (1.2.1)$$

b) in Dempster-Shafer Theory, which is a generalization of Bayesian theory:

$$m_{DST}(A) + m_{DST}(B) + m_{DST}(A \cup B) = 1; \quad (1.2.2)$$

c) in Dezert-Smarandache Theory, which is a generalization of Dempster-Shafer Theory:

$$m_{DSmT}(A) + m_{DSmT}(B) + m_{DSmT}(A \cup B) + m_{DSmT}(A \cap B) = 1; \quad (1.2.3)$$

d) while in the Unification of Fusion Theory, which is a generalization of all of the above:

$$m_{UFT}(A) + m_{UFT}(B) + m_{UFT}(A \cup B) + m_{UFT}(A \cap B) + m_{UFT}(\neg A) + m_{UFT}(\neg B) + m_{UFT}(\neg A \cup \neg B) + m_{UFT}(\neg A \cap \neg B) = 1 \text{ (as V. Christianto observed), where } \neg X \text{ means the negation (or complement) of } X. \quad (1.2.4)$$

The number of terms in the left side of this relationship is equal to:  $2^{(2^n-1)}$  and represents the number of all possible combinations of distinct parts in the Venn diagram, where  $n$  is the cardinal of  $\Theta$ .

Let's consider a frame of discernment  $\Theta$  with exclusive or non-exclusive hypotheses, exhaustive or non-exhaustive frame, closed or open world (so: all possible cases).

We need to make the remark that in case when these  $n \geq 2$  elementary hypotheses  $\theta_1, \theta_2, \dots, \theta_n$  are *exhaustive and exclusive* one gets the Dempster-Shafer Theory, Yager's, Dubois-Prade Theory, while for the case when the hypotheses are *non-exclusive* one gets Dezert-Smarandache Theory, but for *non-exhaustivity* one gets TBM.

An exhaustive frame of discernment is called *close world*, and a non-exhaustive frame of discernment is called *open world* (meaning that new hypotheses might exist in the frame of discernment that we are not aware of).  $\Theta$  may be finite or infinite.

Let  $m_j: S^\Theta \rightarrow [0, 1]$ ,  $1 \leq j \leq s$ , be  $s \geq 2$  basic belief assignments, (1.2.5)

(when bba's are working with crisp numbers),

or with subunitary subsets,  $m_j: S^\Theta \rightarrow P([0, 1])$ , where  $P([0, 1])$  is the set of all subsets of the interval  $[0, 1]$  (when dealing with very imprecise information). (1.2.6)

Normally the sum of crisp masses of a bba,  $m(\cdot)$ , is 1, i.e.  $\sum_{X \in S^\Theta} m(X) = 1$ . (1.2.7)

### 1.3. Incomplete and Paraconsistent Information.

For incomplete information the sum of a bba components can be less than 1 (not enough information known), while in paraconsistent information the sum can exceed 1 (overlapping contradictory information).

The masses can be normalized (i.e. getting the sum of their components = 1), or not (sum of components < 1 in incomplete information; or > 1 in paraconsistent information).

For a bba valued on subunitary subsets one can consider, as normalization of  $m(\cdot)$ ,

either  $\sum_{X \in S^\Theta} \sup \{m(X)\} = 1$ , (1.3.1)

or that there exist crisp numbers  $x \in X$  for each  $X \in S^\Theta$  such that

$$\sum_{\substack{X \in S^{\wedge T} \\ x \in X}} m(x) = 1. \quad (1.3.2)$$

Similarly, for a bba  $m(\cdot)$  valued on subunitary subsets dealing with paraconsistent and incomplete information respectively:

$$\text{a) for incomplete information, one has } \sum_{X \in S^{\wedge T}} \sup \{m(X)\} < 1, \quad (1.3.3)$$

b) while for paraconsistent information one has

$$\sum_{X \in S^{\wedge T}} \sup \{m(X)\} > 1 \quad (1.3.4)$$

and there do not exist crisp numbers  $x \in X$  for each  $X \in S^{\ominus}$  such that  $\sum_{\substack{X \in S^{\wedge T} \\ x \in X}} m(x) = 1.$

#### 1.4. Specificity Chains.

We use the min principle and the precocious/prudent way of computing and transferring the conflicting mass.

Normally by transferring the conflicting mass and by normalization we diminish the specificity. If  $A \cap B$  is empty, then the mass is moved to a less specific element  $A$  (also to  $B$ ), but if we have a pessimistic view on  $A$  and  $B$  we move the mass  $m(A \cap B)$  to  $A \cup B$  (entropy increases, imprecision increases), and even more if we are very pessimistic about  $A$  and  $B$ : we move the conflicting mass to the total ignorance in a closed world, or to the empty set in an open world.

Specificity Chains:

a) From specific to less and less specific (in a closed world):

$$(A \cap B) \subset A \subset (A \cup B) \subset I \text{ or } (A \cap B) \subset B \subset (A \cup B) \subset I. \quad (1.4.1)$$

Also from specific to unknown (in an open world):

$$A \cap B \rightarrow \phi. \quad (1.4.2)$$

b) And similarly for intersections of more elements:  $A \cap B \cap C$ , etc.

$$A \cap B \cap C \subset A \cap B \subset A \subset (A \cup B) \subset (A \cup B \cup C) \subset I \quad (1.4.3)$$

$$\text{or } (A \cap B \cap C) \subset (B \cap C) \subset B \subset (A \cup B) \subset (A \cup B \cup C) \subset I, \text{ etc. in a closed world.} \quad (1.4.4)$$

$$\text{Or } A \cap B \cap C \rightarrow \phi \text{ in an open world.} \quad (1.4.5)$$

c) Also in a closed world:

$$A \cap (B \cup C) \subset B \cup C \subset (B \cup C) \subset (A \cup B \cup C) \subset I \quad \text{or} \quad A \cap (B \cup C) \subset A \subset (A \cup B) \subset (A \cup B \cup C) \subset I. \quad (1.4.6)$$

$$\text{Or } A \cap (B \cup C) \rightarrow \phi \text{ in an open world.} \quad (1.4.7)$$

### 1.5. Static and Dynamic Fusion.

According to Wu Li from NASA we have the following classification and definitions:

*Static fusion* means to combine all belief functions simultaneously.

*Dynamic fusion* means that the belief functions become available one after another sequentially, and the current belief function is updated by combining itself with a newly available belief function.

### 1.6. Unification of Fusion Rules (UFR):

If a variable  $y$  is *directly proportional* with variable  $p$ , then  $y = k_1 \cdot p$ , where  $k_1 \neq 0$  is a constant.

If a variable  $y$  is *inversely proportional* with variable  $q$ , then  $y = k_2 \cdot (1/q)$ , where  $k_2 \neq 0$  is a constant; we can also say herein that  $y$  is directly proportional with variable  $1/q$ .

In a general way, we say that if  $y$  is directly proportional with variables  $p_1, p_2, \dots, p_m$  and inversely proportionally with variables  $q_1, q_2, \dots, q_n$ , then

$$y = k \cdot (p_1 \cdot p_2 \cdot \dots \cdot p_m) / (q_1 \cdot q_2 \cdot \dots \cdot q_n) = k \cdot P / Q, \text{ where } P = \prod_{i=1}^m p_i, Q = \prod_{j=1}^n q_j, \text{ and } k \neq 0 \text{ is a constant.} \quad (1.6.1)$$

With such notations we define a general UFR rule:

$m_{\text{UFR}}(\phi) = 0$ , and  $\forall A \in S^\Theta \setminus \phi$  one has

$$m_{\text{UFR}}(A) = \sum_{\substack{X1, X2 \in S^\Theta \\ X1 * X2 = A}} d(X1 * X2) T(X1, X2) + \frac{P(A)}{Q(A)} \sum_{\substack{X \in S^\Theta \setminus A \\ X * A \in Tr}} d(X * A) \frac{T(A, X)}{P(A) / Q(A) + P(X) / Q(X)} \quad (1.6.2)$$

where  $*$  is an intersection or a union of sets,

$d(X * Y)$  is the degree of intersection or union,

$T(X, Y)$  is a T-norm fusion combination rule (extension of conjunctive or disjunctive rules),

Tr is the ensemble of sets (in majority cases they are empty sets) whose masses must be transferred,

$P(A)$  is the product of all parameters directly proportional with  $A$ ,

while  $Q(A)$  the product of all parameters inversely proportional with  $A$ .

### 1.7. Scenario of Unification of Fusion Theories.

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1.7.1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

1.7.2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

1.7.3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that  $X_1 \vee X_2 \vee \dots \vee X_n$  means either  $X_1$  is reliable, or  $X_2$ , or and so on or  $X_n$ , but not two or more in the same time.

1.7.4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth.

The mixed formula becomes:

$m_{\cap}(\phi) = 0$ , and  $\forall A \in S^{\Theta} \setminus \phi$ ,

$$\text{one has } m_{\cap}(A) = \sum_{\substack{X_1, X_2, X_3, X_4 \in S^{\Theta} \\ ((X_1 \cap X_2) \cup X_3) \cap X_4 = A}} m_1(X_1) m_2(X_2) m_3(X_3) m_4(X_4). \quad (1.7.4.1)$$

1.7.5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes vacuum bba (i.e.  $m(\Theta) = 1$ , and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

1.7.6) If all sources are reliable, or the unreliable sources have been discounted (in the default case), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra  $(\Theta, \cup, \cap, C)$ , no matter what contradictions (or model) the problem has. I emphasize that the super-power set  $S^{\Theta}$  generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

1.7.7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

1.7.8) In the case the model is not known (the default case), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on  $S^{\Theta}$ , and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

1.7.9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then :

1.7.9.1) If an intersection  $A \cap B$  is not empty, we keep the mass  $m(A \cap B)$  on  $A \cap B$ , which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

1.7.9.2) If the intersection  $A \cap B = \emptyset$  is empty, meaning contradiction, we do the following :

1.7.9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass  $m(A \cap B)$  to  $m(A \cup B)$ , since  $A \cup B$  means at least one is right [here one gets *Yager's* if  $n=2$ , or *Dubois-Prade*, or *DSmT*];

1.7.9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass  $m(A \cap B)$  to hypothesis A (nothing is transferred to B);

1.7.9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass  $m(A \cap B)$  to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2-5*]

1.7.9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass  $m(A \cap B)$  to  $A \cup B$  (the more pessimistic the further one gets in the Specificity Chains:  $(A \cap B) \subset A \subset (A \cup B) \subset I$ ); this is also the default case [using *DP's*, *DSm hybrid rule*, *Yager's*];

if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass  $m(A \cap B)$  to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

1.7.9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass  $m(A \cap B)$  to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) - different from A, B,  $A \cup B$  - for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a **closed world** [*DSm hybrid rule*]; but, if it is an **open world**, we can transfer the mass  $m(A \cap B)$  to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

1.7.9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e.  $n = 2$  is the size of the frame of discernment), then one considers the **open world** and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].



Of course, this procedure is extended for any intersections of two or more sets:  $A \cap B \cap C$ , etc. and even for mixed sets:  $A \cap (B \cup C)$ , etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of **decaying the old sources** because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is  $< 1$  as in incomplete information, or  $> 1$  as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule.

## 1.7. Examples:

### 1.7.1. Bayesian Example:

Let  $\Theta = \{A=\text{military motorcycle}, B=\text{civilian motorcycle}, C=\text{unknown type of motorcycle}, D=\text{tank}, E=\text{civilian bicycle with engine}\}$  be the frame of discernment.

	A	B	C	D	E	$A \cap B$	$A \cap C$	$A \cap D$	$A \cap E$	$B \cap C$	$B \cap D$
						$\neq$ $\phi$	$=$ $\phi$	$=$ $\phi$	$=$ $\phi$	Not known if $=$ or $\neq \phi$	$=$ $\phi$
						Consensus between A and B	Contradiction between A and C, but optimistic in both of them	One right, one wrong, but don't know which one	A is right, E is wrong	Don't know the exact model	Unknown any relation between B and D.
$m_1$	0.2	0	0.3	0.4	0.1						
$m_2$	0.5	0.2	0.1	0	0.2						
$m_{12}$	0.10	0	0.03	0	0.02	0.04	0.17	0.20	0.09	0.06	0.08
						↓	↓ A, C	↓	↓ A	↓	↓

						$A \cap B$		$A \cup B$		$B \cap C$ We keep the mass 0.06 on $B \cap C$ till we find out more information on the model.	$B \cup D$
$m_r$						0.04	0.107, 0.063	0.20	0.09	0.06	0.08
<b><math>m_{UFT}</math></b>	<b>0.324</b>	<b>0.040</b>	<b>0.119</b>	<b>0</b>	<b>0.027</b>	<b>0.04</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0.06</b>	<b>0</b>
$m_{lower}$ (closed world)	0.10	0	0.03	0	0.02						
$m_{lower}$ (open world)	0.10	0	0.03	0	0.02						
$m_{middle}$ (default)	0.10	0	0.03	0	0.02						
$m_{upper}$	0.400	0.084	0.178	0.227	0.111						

Table 1.7.1.1 (Part 1). Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

	$B \cap E$	$C \cap D$	$C \cap E$	$D \cap E$	$A \cup B$	$A \cup C$	$A \cup D$	$A \cup E$	$B \cup C$
	$\neq$ $\phi$	$=$ $\phi$	$=$ $\phi$	$=$ $\phi$					
	The intersection is not empty, but neither $B \cap E$ nor $B \cup E$ interest us	Pessimistic in both C and D	Very pessimistic in both C and E	Both D and E are wrong					
$m_1$									
$m_2$									
$m_{12}$	0.02	0.04	0.07	0.08					
	$\downarrow$ B, E	$\downarrow$ $C \cup D$	$\downarrow$ $A \cup B \cup C \cup D \cup E$	$\downarrow$ A, B, C					
$m_r$	0.013, 0.007	0.04	0.07	0.027, 0.027, 0.027					
<b><math>m_{UFT}</math></b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0.20</b>	<b>0</b>	<b>0</b>
$m_{lower}$ (closed world)									
$m_{lower}$ (open world)									
$m_{middle}$ (default)					0.04	0.17	0.20	0.09	0.06
$m_{upper}$									

Table 1.7.1.1 (Part 2). Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

	B $\cup$ D	B $\cup$ E	C $\cup$ D	C $\cup$ E	D $\cup$ E	A $\cup$ B $\cup$ C $\cup$ D $\cup$ E	$\phi$
m <sub>1</sub>							
m <sub>2</sub>							
m <sub>12</sub>							
m <sub>f</sub>							
<b>m<sub>UFT</sub></b>	<b>0.08</b>	<b>0</b>	<b>0.04</b>	<b>0</b>	<b>0</b>	<b>0.07</b>	<b>0</b>
m <sub>lower</sub> (closed world)						0.85	
m <sub>lower</sub> (open world)							0.85
m <sub>middle</sub> (default)	0.08	0.02	0.04	0.07	0.08		
m <sub>upper</sub>							

Table 1.7.1.1 (Part 3). Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

We keep the mass  $m_{12}(B \cap C) = 0.06$  on  $B \cap C$  (eleventh column in Table 1.7.1.1, Part 1) although we don't know if the intersection  $B \cap C$  is empty or not (this is considered the default model), since in the case when it is empty one considers an open world because  $m_{12}(\phi) = 0.06$  meaning that there might be new possible hypotheses in the frame of discernment, but if  $B \cap C \neq \phi$  one considers a consensus between B and C.

Later, when finding out more information about the relation between B and C, one can transfer the mass 0.06 to  $B \cup C$ , or to the total ignorance I, or split it between the elements B, C, or even keep it on  $B \cap C$ .

$m_{12}(A \cap C) = 0.17$  is redistributed to A and C using the PCR5:

$$a_1/0.2 = c_1/0.1 = 0.02(0.2+0.1),$$

$$\text{whence } a_1 = 0.2(0.02/0.3) = 0.013,$$

$$c_1 = 0.1(0.02/0.3) = 0.007.$$

$$a_2/0.5 = c_2/0.3 = 0.15(0.5+0.3),$$

$$\text{whence } a_2 = 0.5(0.15/0.8) = 0.094,$$

$$c_2 = 0.3(0.15/0.8) = 0.056.$$

Thus A gains  $a_1 + a_2 = 0.013 + 0.094 = 0.107$  and C gains  $c_1 + c_2 = 0.007 + 0.056 = 0.063$ .

$m_{12}(B \cap E) = 0.02$  is redistributed to B and E using the PCR5:

$$b/0.2 = e/0.1 = 0.02/(0.2+0.1),$$

$$\text{whence } b = 0.2(0.02/0.3) = 0.013,$$

$$e = 0.1(0.02/0.3) = 0.007.$$

Thus B gains 0.013 and E gains 0.007.

Then one sums the masses of the conjunctive rule  $\mathbf{m}_{12}$  and the redistribution of conflicting masses  $\mathbf{m}_r$  (according to the information we have on each intersection, model, and relationship between conflicting hypotheses) in order to get the mass of the Unification of Fusion Theories rule  $\mathbf{m}_{\text{UFT}}$ .

$\mathbf{m}_{\text{UFT}}$ , the Unification of Fusion Theories rule, is a combination of many rules and gives the optimal redistribution of the conflicting mass for each particular problem, following the given model and relationships between hypotheses; this extra-information allows the choice of the combination rule to be used for each intersection. The algorithm is presented above.

$\mathbf{m}_{\text{lower}}$ , the lower bound believe assignment, the most pessimistic/prudent belief, is obtained by transferring the whole conflicting mass to the total ignorance (Yager's rule) in a closed world, or to the empty set (Smets' TBM) in an open world herein meaning that other hypotheses might belong to the frame of discernment.

$\mathbf{m}_{\text{middle}}$ , the middle believe assignment, half optimistic and half pessimistic, is obtained by transferring the partial conflicting masses  $\mathbf{m}_{12}(X \cap Y)$  to the partial ignorance  $X \cup Y$  (as in Dubois-Prade theory or more general as in Dezert-Smarandache theory).

Another way to compute a middle believe assignment would be to average the  $\mathbf{m}_{\text{lower}}$  and  $\mathbf{m}_{\text{upper}}$ .

$\mathbf{m}_{\text{upper}}$ , the upper bound believe assignment, the most optimistic (less prudent) belief, is obtained by transferring the masses of intersections (empty or non-empty) to the elements in the frame of discernment using the PCR5 rule of combination, i.e.  $\mathbf{m}_{12}(X \cap Y)$  is split to the elements X, Y (see Table 2). We use PCR5 because it is more exact mathematically (following backwards the tracks of the conjunctive rule) than Dempster's rule, minC, and PCR1-4.

X	$\mathbf{m}_{12}(X)$	A	B	C	D	E
$A \cap B$	0.040	0.020	0.020			
$A \cap C$	0.170	0.107		0.063		
$A \cap D$	0.200	0.111			0.089	
$A \cap E$	0.090	0.020 0.042				0.020 0.008
$B \cap C$	0.060		0.024	0.036		
$B \cap D$	0.080		0.027		0.053	

$B \cap E$	0.020		0.013			0.007
$C \cap D$	0.040			0.008	0.032	
$C \cap E$	0.070			0.036 0.005		0.024 0.005
$D \cap E$	0.080				0.053	0.027
Total	0.850	0.300	0.084	0.148	0.227	0.091

Table 1.7.1.2. Redistribution of the intersection masses to the singletons A, B, C, D, E using the PCR5 rule only, needed to compute the upper bound belief assignment  $m_{upper}$ .

### 1.7.2. Negation/Complement Example:

Let  $\Theta = \{A=\text{aircraft}, B=\text{terrestrial vehicle}, C=\text{military car}, D=\text{enemy object}\}$  be the frame of discernment. Since  $(\Theta, \cup, \cap, C)$  is Boolean algebra, the super-power set  $S^\Theta$  includes complements/negations, intersections and unions. Let's note by  $C(B)$  the complement of B.

	A	B	D	$C(B)$	$A \cap C$ {Later in the dynamic fusion process we find out that $A \cap C$ is empty }	$B \cup C = B$	$A \cap B$	$A \cap C(B) = A$
					$=$ $\phi$		$=$ $\phi$	$\neq$ $\phi$
					Unknown relationship between A and C		Optimistic in both A and B.	Consensus between A and $C(B)$ , but $A \subset$ $C(B)$
$m_1$	0.2	0.3	0	0.1	0.1	0.3		
$m_2$	0.4	0.1	0	0.2	0.2	0.1		
$m_{12}$	0.08	0.09	0	0.02	0.17	0.03	0.14	0.08
					$\downarrow$ $A \cup C$	$\downarrow$ B	$\downarrow$ A, B	$\downarrow$ A
$m_r$					0.17	0.03	0.082, 0.058	0.08
<b><math>m_{UFT}</math></b>	<b>0.277</b>	<b>0.318</b>	<b>0.035</b>	<b>0.020</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>
$m_{lower}$ (closed world)	0.16	0.26	0	0.02	0	0		
$m_{lower}$ (open world)	0.16	0.26	0	0.02	0	0		
$m_{middle}$ (default)	0.16	0.26	0	0.02	0	0		
$m_{upper}$	0.296	0.230	0	0.126	0.219	0.129		

Table 1.7.2.1 (Part 1). Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

	$A \cap (B \cup C)$	$B \cap C(B)$	$B \cap (A \cap C)$	$C(B) \cap (A \cap C)$	$C(B) \cap (B \cup C) =$ $C(B) \cap C$	$B \cup (A \cap C) = B$
	$=$ $\phi$	$=$ $\phi$	$=$ $\phi$	$=$ $\phi$	$=$ $\phi$	
	At least one is right between A	B is right, $C(B)$ is	No relationship known between B	Very pessimistic on $C(B)$ and	Neither $C(B)$ nor $B \cup C$ are right	

	and $B \cup C$	wrong	and $A \cap C$ (default case)	$A \cap C$		
$m_1$						
$m_2$						
$m_{12}$	0.14	0.07	0.07	0.04	0.07	
	$\downarrow$ $A \cup (B \cup C)$	$\downarrow$ $B$	$\downarrow$ $B \cup (A \cap C) = B$	$\downarrow$ $A \cup B \cup C \cup D$	$\downarrow$ $A, D$	$\downarrow$ $B$
$m_r$	0.14	0.07	0.07	0.04	0.035, 0.035	
<b><math>m_{UFT}</math></b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>
$m_{lower}$ (closed world)						
$m_{lower}$ (open world)						
$m_{middle}$ (default)						
$m_{upper}$						

Table 1.7.2.1 (Part 2). Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

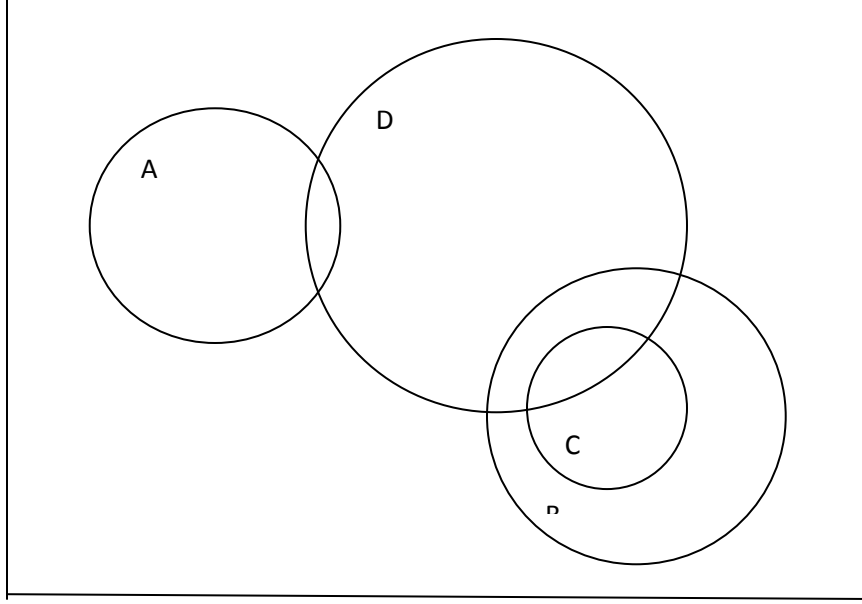
	$A \cup B$	$A \cup C$	$A \cup D$	$B \cup C$	$B \cup D$	$C \cup D$	$A \cup B \cup C$	$A \cup B \cup C \cup D$	$\phi$
$m_1$									
$m_2$									
$m_{12}$									
$m_r$									
<b><math>m_{UFT}</math></b>	<b>0</b>	<b>0.170</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>0.140</b>	<b>0.040</b>	<b>0</b>
$m_{lower}$ (closed world)								0.56	
$m_{lower}$ (open world)									0.56
$m_{middle}$ (default)	0.14	0.17		0.03			0.14	0.11	
$m_{upper}$									

Table 1.7.2.1 (Part 3). Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses.

#### Model of Negation/Complement Example:

$$A \cap B = \phi, C \subset B, A \subset C(B).$$

Fig. 1.7.2.1:



$$m_{12}(A \cap B) = 0.14.$$

$$x_1/0.2 = y_1/0.1 = 0.02/0.3, \text{ whence } x_1 = 0.2(0.02/0.3) = 0.013, y_1 = 0.1(0.02/0.3) = 0.007;$$

$$x_2/0.4 = y_2/0.3 = 0.12/0.7, \text{ whence } x_2 = 0.4(0.12/0.7) = 0.069, y_2 = 0.3(0.12/0.7) = 0.051.$$

Thus, A gains  $0.013 + 0.069 = 0.082$  and B gains  $0.007 + 0.051 = 0.058$ .

For the upper belief assignment  $m_{\text{upper}}$  one considered all resulted intersections from results of the conjunctive rule as empty and one transferred the partial conflicting masses to the elements involved in the conflict using PCR5.

All elements in the frame of discernment were considered non-empty.

### 1.7.3. Example with Intersection:

Look at this:

Suppose we have three geographical regions of a military battle field whose x-coordinates are  $A = \{x < 0.4\}$ ,  $B = \{0.3 < x < 0.6\}$ ,  $C = \{x > 0.8\}$ . The frame of discernment  $T = \{A, B, C\}$  represents the possible cross section of a military target, and there are two sensors giving the following bba's:

$$m_1(A) = 0.5, m_1(B) = 0.2, m_1(C) = 0.3.$$

$$m_2(A) = 0.4, m_2(B) = 0.4, m_2(C) = 0.2.$$

	A	B	C	$A \cap B = \{.3 < x < .4\}$	$A \cup C$	$B \cup C$
$m_1$	.5	.2	.3			
$m_2$	.4	.4	.2			
$m_1 \& m_2$	.20	.08	.06	.28	.22	.16
DSmT						

We have a DSm hybrid model (one intersection  $A \& B = \text{nonempty}$  ).

This example proves the necessity of allowing intersections of elements in the frame of discernment.  
[Shafer's model doesn't apply here.]

Dezert-Smarandache Theory of Uncertain and Paradoxist Reasoning (DSmT) is the only theory which accepts intersections of elements.

#### 1.7.4. Another Multi-Example of UFT:

Cases:

1.7.4.1. Both sources reliable: use conjunctive rule [default case]:

1.7.4.1.1.  $A \cap B \neq \emptyset$ :

1.7.4.1.1.1. Consensus between A and B; mass  $\rightarrow A \cap B$ ;

1.7.4.1.1.2. Neither  $A \cap B$  nor  $A \cup B$  interest us; mass  $\rightarrow A, B$ ;

1.7.4.1.2.  $A \cap B = \emptyset$ :

1.7.4.1.2.1. Contradiction between A and B, but optimistic in both of them; mass  $\rightarrow A, B$ ;

1.7.4.1.2.2. One right, one wrong, but don't know which one; mass  $\rightarrow A \cup B$ ;

1.7.4.1.2.3. Unknown any relation between A and B [default case]; mass  $\rightarrow A \cup B$ ;

1.7.4.1.2.4. Pessimistic in both A and B; mass  $\rightarrow A \cup B$ ;

1.7.4.1.2.5. Very pessimistic in both A and B;

1.7.4.1.2.5.1. Total ignorance  $\supset A \cup B$ ; mass  $\rightarrow A \cup B \cup C \cup D$  (total ignorance);

1.7.4.1.2.5.2. Total ignorance =  $A \cup B$ ; mass  $\rightarrow \emptyset$  (open world);



1.7.4.1.2.6. A is right, B is wrong; mass  $\rightarrow$  A;

1.7.4.1.2.7. Both A and B are wrong; mass  $\rightarrow$  C, D;

1.7.4.1.3. Don't know if  $A \cap B =$  or  $\neq \phi$  (don't know the exact model); mass  $\rightarrow A \cap B$  (keep the mass on intersection till we find out more info) [default case];

1.7.4.2. One source reliable, other not, but not known which one: use disjunctive rule; no normalization needed.

1.7.4.3. S1 reliable, S2 not reliable 20%: discount S2 for 20% and use conjunctive rule.

	A	B	$A \cup B$	$A \cap B$	$\phi$ (open world)	$A \cup B \cup C \cup D$	C	D
S1	.2	.5	.3					
S2	.4	.4	.2					
S1&S2	.24	.42	.06	.28				
S1 or S2	.08	.20	.72	0				
UFT 1.7.4.1.1.1	.24	.42	.06	.28				
UFT 1.7.4.1.1.2 (PCR5)	.356	.584	.060	0				
UFT 1.7.4.1.2.1	.356	.584	.060	0				
UFT 1.7.4.1.2.2	.24	.42	.34	0				
UFT 1.7.4.1.2.3	.24	.42	.34	0				
UFT 1.7.4.1.2.4	.24	.42	.34	0				
UFT 1.7.4.1.2.5.1	.24	.42	.06	0	0	.28		
UFT 1.7.4.1.2.5.2	.24	.42	.06	0	.28			
80% S2	.32	.32	.16			.20		
UFT 1.7.4.1.2.6	.52	.42	.06					
UFT 1.7.4.1.2.7	.24	.42	.06	0			.14	.14

UFT 1.7.4.1.3	.24	.42	.06	.28				
UFT 1.7.4.2	.08	.20	.72	0				
UFT 1.7.4.3	.232	.436	.108	.224		0		

## 2) Unification of Filter Algorithms (UFA)

The idea is to extend the NIS [Normalized Innovation Squared] procedure and take into consideration not only the Kalman Filter (KF), Extended Kalman Filter (EKF), Unscented Kalman Filter (UKF), and Particle Filter (PF), but all filtering methods and algorithms - and use, depending on the military application and its computing complexity, the one that fits the best. At each step UFA should check what filter works better and use that filter.

Include linear and non-linear filters:

- Kalman Filter (KF);
- Extended Kalman Filter (EKF);
- Unscented Kalman Filter (UKF);
- Particle Filter (PF);
- Daum Filter;
- Alpha-Beta Filter;
- Alpha-Beta-Gamma Filter;
- Weiner Filter;

etc.

## 3) Data Fusion and Image Fusion Methods using Neutrosophic Logic

Firstly we recall the definitions of T-norm and T-conorm from fuzzy logic and set, and then we present their generalizations to N-norm and N-conorm from neutrosophic logic and set.

### 3.1) T-norm and T-conorm.

Defining the T-norm conjunctive consensus:

The t-norm conjunctive consensus is based on the particular t-norm function. In general it is a function defined in fuzzy set/logic theory in order to represent the intersection between two particular fuzzy sets. If one extends T-norm to the data fusion theory, it will be a substitute for the conjunctive rule.

The T-norm has to satisfy the following conditions:

- Associativity:

$$T_{norm}(T_{norm}(x, y), z) = T_{norm}(x, T_{norm}(y, z)) \quad (3.1.1)$$

- Commutativity

$$T_{norm}(x, y) = T_{norm}(y, x) \quad (3.1.2)$$

- Monotonicity

$$\text{if } (x \leq a) \& (y \leq b) \text{ then } T_{norm}(x, y) \leq T_{norm}(a, b) \quad (3.1.3)$$

- Boundary Conditions:

$$T_{norm}(x, 0) = 0; \quad T_{norm}(x, 1) = x \quad (3.1.4)$$

There are many functions which satisfy these T-norm conditions:

- Zadeh's (default) min operator:

$$m(X) = \min \{m_1(X_i), m_2(X_j)\} \quad (3.1.5)$$

- Algebraic product operator:

$$m(X) = m_1(X_i) \cdot m_2(X_j) \quad (3.1.6)$$

- Bounded product operator:

$$m(X) = \max \{0, m_1(X_i) + m_2(X_j) - 1\} \quad (3.1.7)$$

Defining the T-conorm disjunctive consensus:

The t-conorm disjunctive consensus is based on the particular t-conorm function. In general it is a function defined in fuzzy set/logic theory in order to represent the union between two particular fuzzy sets. If one extends T-conorm to the data fusion theory, it will be a substitute for the disjunctive rule.

The T-conorm has to satisfy the following conditions:

- Associativity:

$$T_{conorm}(T_{conorm}(x, y), z) = T_{conorm}(x, T_{conorm}(y, z)) \quad (3.1.8)$$

- Commutativity

$$T_{conorm}(x, y) = T_{conorm}(y, x) \quad (3.1.9)$$

- Monotonicity

$$\text{if } (x \leq a) \& (y \leq b) \text{ then } T_{conorm}(x, y) \leq T_{conorm}(a, b) \quad (3.1.10)$$

- Boundary Conditions:

$$T_{conorm}(x, 0) = x; \quad T_{conorm}(x, 1) = 1 \quad (3.1.11)$$

There are many functions which satisfy these T-norm conditions:

- Zadeh's (default) max operator:

$$m(X) = \max \{m_1(X_i), m_2(X_j)\} \quad (3.1.12)$$

- Algebraic product operator:

$$m(X) = m_1(X_i) + m_2(X_j) - m_1(X_i) \cdot m_2(X_j) \quad (3.1.13)$$

- Bounded product operator:

$$m(X) = \min \{m_1(X_i) + m_2(X_j), 1\} \quad (3.1.14)$$

The way of association between the focal elements of the given two sources of information,  $m_1(\cdot)$  and  $m_2(\cdot)$ , is defined as  $X = \theta_i \cap \theta_j$  and the degree of association is as follows:

$$\tilde{m}_{12}(X) = T_{norm} \{m_1(\theta_i) + m_2(\theta_j)\} \quad (3.1.15)$$

where  $\tilde{m}_{12}(X)$  represents the basic belief assignments (bba) after the fusion, associated with the given proposition  $X$  by using particular t-norm based conjunctive rule.

**Step 2:** Distribution of the mass, assigned to the conflict.

In some degree it follows the distribution of conflicting mass in the most sophisticated DSMT based Proportional Conflict Redistribution rule number 5, but the procedure here relies on fuzzy operators.

The total conflicting mass is distributed to all non-empty sets proportionally with respect to the *Maximum (Sum)* between the elements of corresponding mass matrix's columns, associated with the given element  $X$  of the power set. It means the bigger mass is redistributed towards the element, involved in the conflict and contributing to the conflict with the maximum specified probability mass.

The general procedure for fuzzy based conflict redistribution is as follows:

- Calculate all partial conflict masses separately;
- If  $\theta_i \cap \theta_j = \emptyset$ , then  $\theta_i$  and  $\theta_j$  are involved in the conflict; redistribute the corresponding

masses  $m_{12}(\theta_i \cap \theta_j) > 0$ , involved in the particular partial conflicts to the non-empty sets  $\theta_i$  and  $\theta_j$  with respect to  $\max \{m_1(\theta_i), m_2(\theta_j)\}$  and with respect to  $\max \{m_1(\theta_j), m_2(\theta_i)\}$ ;

- Finally, for the given above two sources, the  $T_{norm}$  conjunctive consensus yields:

$$\tilde{m}_{12}(\theta_i) = T_{norm}(m_1(\theta_i), m_2(\theta_j)) + T_{norm}(m_1(\theta_i), m_2(\theta_i \cup \theta_j)) + T_{norm}(m_1(\theta_i \cup \theta_j), m_2(\theta_i)) \quad (3.1.16)$$

$$\tilde{m}_{12}(\theta_j) = T_{norm}(m_1(\theta_j), m_2(\theta_i)) + T_{norm}(m_1(\theta_j), m_2(\theta_i \cup \theta_j)) + T_{norm}(m_1(\theta_i \cup \theta_j), m_2(\theta_j)) \quad (3.1.17)$$

$$\tilde{m}_{12}(\theta_i \cup \theta_j) = T_{norm}(m_1(\theta_i \cup \theta_j), m_2(\theta_i \cup \theta_j)) \quad (3.1.18)$$

**Step 3:** The basic belief assignment, obtained as a result of the applied TCN rule becomes:

$$\tilde{m}_{PCR5}^{TCN}(\theta_i) = \tilde{m}_{12}(\theta_i) + m_1(\theta_i) \times \frac{T_{norm}(m_1(\theta_i), m_2(\theta_j))}{T_{conorm}(m_1(\theta_i), m_2(\theta_j))} + m_2(\theta_i) \times \frac{T_{norm}(m_1(\theta_j), m_2(\theta_i))}{T_{conorm}(m_1(\theta_j), m_2(\theta_i))}$$

(3.1.19)

$$\tilde{m}_{PCR}^{TCN}(\theta_j) = \tilde{m}_{12}(\theta_j) + m_2(\theta_j) \times \frac{T_{norm}(m_1(\theta_i), m_2(\theta_j))}{T_{conorm}(m_1(\theta_i), m_2(\theta_j))} + m_1(\theta_i) \times \frac{T_{norm}(m_1(\theta_j), m_2(\theta_i))}{T_{conorm}(m_1(\theta_j), m_2(\theta_i))} \quad (3.1.20)$$

**Step 4:** Normalization of the result.

The final step of the TCN fusion rule concerns the normalization procedure:

$$\tilde{m}_{PCR5}^{TCN}(\theta) = \frac{\tilde{m}_{TCN}(\theta)}{\sum_{\substack{\theta \neq \emptyset \\ \theta \in 2^\Theta}} \tilde{m}_{TCN}(\theta)} \quad (3.1.21)$$

This  $\tilde{m}_{PCR5}^{TCN}(\cdot)$  is good, but it is a deviation from *PCR5* principle of distribution of the conflicting information, because it divides by  $\max\{m_1(A), m_2(X)\}$  and respectively by  $\max\{m_2(A), m_1(X)\}$ . It also substitutes the conjunctive rule with the min operator.

Let's suppose  $A \cap B = \emptyset$ , and we have:

	A	B	$A \cup B$	$A \cap B$
$m_1$	0.3	0	0.7	
$m_2$	0	0.6	0.4	

We use the min operator for the conjunctive rule and we get:

min	0.3	0.6	0.4	0.3
-----	-----	-----	-----	-----

We need to transfer  $m_{\min}(A \cap B) = 0.3$  to  $A$  and  $B$  proportionally to their masses 0.3 and respectively 0.6.

$$\frac{x}{0.3} = \frac{y}{0.6} = \frac{\min\{0.3, 0.6\}}{0.3 + 0.6} = \frac{0.3}{0.9} = \frac{1}{3},$$

so,

$$x = 0.3 \left( \frac{1}{3} \right) = 0.1, \quad y = 0.6 \left( \frac{1}{3} \right) = 0.2$$

But at this step it uses the max operator and divides by  $\max\{\dots\}$ .

One gets:

$$x = (0.3) \frac{\min\{0.3, 0.6\}}{\max\{0.3, 0.6\}} = 0.3 \cdot \frac{0.3}{0.6} = 0.15$$

$$y = (0.6) \frac{\min\{0.3, 0.6\}}{\max\{0.3, 0.6\}} = 0.6 \cdot \frac{0.3}{0.6} = 0.30$$

So, actually the conflict  $\min\{0.3, 0.6\} = 0.3 = m_{\min}(A \cap B)$  is not distributed proportionally to  $A$  and  $B$ .

A better formula (doing a better redistribution of the conflict, exactly as *PCR5*) is:

$$m_{PCR5v2}^{TN}(A) = \sum_{\substack{X, Y \in G^\Theta \\ X \cap Y = A}} \min\{m_1(X), m_2(Y)\} +$$

$$+ \sum_{\substack{X \in G^\Theta \\ X \cap A = \emptyset}} \left[ m_1(A) \cdot \frac{\min\{m_1(A), m_2(X)\}}{m_1(A) + m_2(X)} + m_2(A) \cdot \frac{\min\{m_2(A), m_1(X)\}}{m_2(A) + m_1(X)} \right] \quad (3.1.22)$$

And then we normalize:

$$\tilde{m}_{PCR5v2}^{TN}(A) = \frac{m_{PCR5v2}^{TN}(A)}{\sum_{X \in G^\Theta} m_{PCR5v2}^{TCN}(X)}. \quad (3.1.23)$$

We use the notation  $TN$  (= T-norm) only, since T-conorm is not used.

In a more general way, we can define a class of fuzzy fusion rules based on  $PCR5$ , combining two masses  $m_1(\cdot)$  and  $m_2(\cdot)$ , and corresponding to the conjunctive rule:

$$\begin{aligned} m_{PCR5v2}^{TN_g}(A) = & \sum_{\substack{X, Y \in G^\Theta \\ X \cap Y = A}} TN(m_1(X), m_2(Y)) + \\ & + \sum_{\substack{X \in G^\Theta \\ X \cap A = \emptyset}} \left[ m_1(A) \cdot \frac{TN\{m_1(A), m_2(X)\}}{m_1(A) + m_2(X)} + m_2(A) \cdot \frac{TN\{m_2(A), m_1(X)\}}{m_2(A) + m_1(X)} \right] \end{aligned} \quad (3.1.24)$$

where  $TN(\cdot, \cdot)$  is a fuzzy T-norm. Then we normalize.

If  $TN(\cdot, \cdot)$  is the product, we get just  $PCR5$ .

If  $TN(\cdot, \cdot)$  is the min, we get the previous  $m_{PCR5v2}^{TN}(\cdot)$ .

We can replace  $TN(\cdot, \cdot)$  by other fuzzy T-norms and obtain different fuzzy fusion rules.

The  $TN$  conjunctive rule is used.

$$\text{So, } m_c^{TN}(A) = \sum_{\substack{X, Y \in G^\Theta \\ X \cap Y = A}} TN(m_1(X), m_2(Y)), \quad (3.1.25)$$

$$\text{and then we normalize: } m_D^{TN}(A) = \frac{m_c^{TN}(A)}{\sum_{X \in G^\Theta} m_c^{TN}(X)}, \quad (3.1.26)$$

which is the  $TN$  Dempster's rule.

If  $TN(\cdot, \cdot)$  is the product, we get just Dempster's rule.

$$\text{Similarly, we can define the fuzzy disjunctive rule: } m_c^{TCN}(A) = \sum_{\substack{X, Y \in G^\Theta \\ X \cup Y = A}} TCN(m_1(X), m_2(Y)) \quad (3.1.27)$$

and then we normalize.

Thus, the  $TCN$  Dubois-Prade rule is:

$$m_{DP}^{TCN}(A) = \sum_{\substack{X, Y \in G^\Theta \\ X \cap Y = A}} TN(m_1(X), m_2(Y)) + \sum_{\substack{X, Y \in G^\Theta \\ X \cap Y = \emptyset \\ X \cup Y = A}} TCN(m_1(X), m_2(Y)) \quad (3.1.28)$$

and then we normalize.

Similarly, for DS<sub>SmH</sub>, Yager's rule, Smets' rule, etc.

Let's take the same example showed in the below Open Table 3.1.1:

	<i>F</i>			<i>H</i>		$\emptyset$	$\emptyset$	$\emptyset$
	$O_1$	$O_2$	$O_3$	$O_1 \cup O_3$	$O_1 \cup O_2 \cup O_3$	$O_1 \cap O_2$	$O_2 \cap O_3$	$O_2 \cap (O_1 \cup O_3)$
$m_x$	0.2	0.2	0.3	0	0.3			
$\mu_{T_y}$	0	0.4	0	0.6	0			
$m_{conj.min}$	0.2	0.5	0.3	0.3	0	0.2	0.3	0.2
For <i>TN</i> Dempster's rule we simply normalize the masses of non-empty sets:								
<i>TN</i> Dempster's Rule	$\frac{2}{13}$	$\frac{5}{13}$	$\frac{3}{13}$	$\frac{3}{13}$	0	0	0	0
<i>TCN</i> Dubois- Prade and TCN DSmH non-normalized	0.2	0.5	0.3	0.3	0	$O_1 \cup O_2$ 0.2	$O_2 \cup O_3$ 0.3	$O_2 \cup O_1 \cup O_3$ 0.2
<i>TCN DP</i> and TCN DSmH normalized	0.10	0.25	0.15	0.15	0.10	$O_1 \cup O_2$ 0.10	$O_2 \cup O_3$ 0.15	
$m_{PCR5v2}^{TN}$ non-normalized different form <i>TN</i>	0.267	0.854	0.429	0.450	0	0	0	0

For non-norm  $m_{PCR5v2}^{TN}$  we need to transfer:  $m_{conj.min}(O_1 \cap O_2) = \min\{0.2, 0.4\} = 0.2$  to  $O_1$  and  $O_2$  proportionally with 0.2 and 0.4 respectively (as in *PCR5*):

$$\frac{x_1}{0.2} = \frac{y_1}{0.4} = \frac{0.2}{0.2+0.4} = \frac{0.2}{0.6} = \frac{1}{3}, \text{ so } x_1 = 0.2 \left( \frac{1}{3} \right) = 0.067, \quad y_1 = 0.4 \left( \frac{1}{3} \right) = 0.133.$$

Similarly,  $m_{conj.min}(O_2 \cap O_3) = \min\{0.4, 0.3\} = 0.3$  should be transferred to  $O_2$  and  $O_3$ :

$$\frac{y_2}{0.4} = \frac{z_2}{0.3} = \frac{0.3}{0.4+0.3} = \frac{0.3}{0.7} = \frac{3}{7}; \text{ so } y_2 = 0.4 \left( \frac{3}{7} \right) = 0.171, \quad z_2 = 0.3 \left( \frac{3}{7} \right) = 0.129.$$

Same for  $m_{conj.min}(O_2 \cap (O_1 \cup O_3)) = \min\{0.2, 0.6\} = 0.2$ , which should be transferred to

$$O_2 \text{ and } O_1 \cup O_3: \frac{y_3}{0.2} = \frac{w_3}{0.6} = \frac{0.2}{0.2+0.6} = \frac{0.2}{0.8} = \frac{1}{4}, \text{ so } y_3 = 0.2 \left( \frac{1}{4} \right) = 0.05,$$

$$w_3 = 0.6 \left( \frac{1}{4} \right) = 0.15$$

$m_{PCR5v2}^{TN}$ normalized different form Hebena's	0.1335	0.4270	0.2145	0.2250	0	0	0	0
<i>TN</i> Yager's rule not normalized	0.2	0.5	0.3	0.3	0.7			
<i>TN</i> Yager's rule normalized	0.10	0.25	0.15	0.15	0.35			

$TN$ <i>Smet's rule not normalized</i>	0.2	0.5	0.3	0.3	0	0.7
					$\emptyset$	
$TN$ <i>Smet's rule normalized</i>	0.10	0.25	0.15	0.15	0	0.35
					$\emptyset$	

We'd define the  $m_{PCR5}^{TCN}$  rule (with respect to  $PCR5$ ) in the following condensed way:

$$m_{PCR5}^{TCN}(A) = \sum_{\substack{X, Y \in G^{\Theta} \\ X \cap Y = A}} \min\{m_1(X), m_2(Y)\} + \\ + \sum_{\substack{X \in G^{\Theta} \\ X \cap A = \emptyset}} \left[ m_1(A) \cdot \frac{\min\{m_1(A), m_2(X)\}}{\max\{m_1(A) + m_2(X)\}} + m_2(A) \cdot \frac{\min\{m_2(A), m_1(X)\}}{\max\{m_2(A) + m_1(X)\}} \right]$$

And then we normalize.

The first sum is the conjunctive rule using the min operator.

We can extend these min/max operators to many other fusion rules: DSmH, Dempster's rule, disjunctive rule, Dubois-Prade's, Yager's, etc.

### 3.3. Definition of the Neutrosophic Logic/Set:

Let  $T, I, F$  be real standard or non-standard subsets of  $]0, 1[^+$ ,  
with  $\sup T = t\_sup$ ,  $\inf T = t\_inf$ ,  
 $\sup I = i\_sup$ ,  $\inf I = i\_inf$ ,  
 $\sup F = f\_sup$ ,  $\inf F = f\_inf$ ,  
and  $n\_sup = t\_sup + i\_sup + f\_sup$ ,  
 $n\_inf = t\_inf + i\_inf + f\_inf$ .

Let  $U$  be a universe of discourse, and  $M$  a set included in  $U$ . An element  $x$  from  $U$  is noted with respect to the set  $M$  as  $x(T, I, F)$  and belongs to  $M$  in the following way: it is  $t\%$  true in the set,  $i\%$  indeterminate (unknown if it is or not) in the set, and  $f\%$  false, where  $t$  varies in  $T$ ,  $i$  varies in  $I$ ,  $f$  varies in  $F$ .

Statically  $T, I, F$  are subsets, but dynamically  $T, I, F$  are functions/operators depending on many known or unknown parameters.

### 3.4. In a similar way define the Neutrosophic Logic:

A logic in which each proposition  $x$  is  $T\%$  true,  $I\%$  indeterminate, and  $F\%$  false, and we write it  $x(T, I, F)$ , where  $T, I, F$  are defined above.

### 3.5. As a generalization of T-norm and T-conorm from the Fuzzy Logic and Set, we now introduce the **N-norms and N-conorms** for the Neutrosophic Logic and Set.

We define a *partial relation order* on the neutrosophic set/logic in the following way:  
 $x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2)$  iff (if and only if)  $T_1 \leq T_2$ ,  $I_1 \geq I_2$ ,  $F_1 \geq F_2$  for crisp components.



And, in general, for subunitary set components:

$$x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2) \text{ iff}$$

$$\begin{aligned} \inf T_1 &\leq \inf T_2, \sup T_1 \leq \sup T_2, \\ \inf I_1 &\geq \inf I_2, \sup I_1 \geq \sup I_2, \\ \inf F_1 &\geq \inf F_2, \sup F_1 \geq \sup F_2. \end{aligned}$$

If we have mixed - crisp and subunitary - components, or only crisp components, we can transform any crisp component, say “a” with  $a \in [0,1]$  or  $a \in ]0, 1^+[$ , into a subunitary set  $[a, a]$ . So, the definitions for subunitary set components should work in any case.

### 3.6. N-norms

$$N_n: (]0,1^+[ \times ]0,1^+[ \times ]0,1^+[ )^2 \rightarrow ]0,1^+[ \times ]0,1^+[ \times ]0,1^+[$$

$$N_n(x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = (N_nT(x, y), N_nI(x, y), N_nF(x, y)),$$

where  $N_nT(\cdot, \cdot)$ ,  $N_nI(\cdot, \cdot)$ ,  $N_nF(\cdot, \cdot)$  are the truth/membership, indeterminacy, and respectively falsehood/nonmembership components. (3.6.1)

$N_n$  have to satisfy, for any  $x, y, z$  in the neutrosophic logic/set  $M$  of the universe of discourse  $U$ , the following axioms:

a) Boundary Conditions:  $N_n(x, \mathbf{0}) = \mathbf{0}$ ,  $N_n(x, \mathbf{1}) = x$ . (3.6.2)

b) Commutativity:  $N_n(x, y) = N_n(y, x)$ . (3.6.3)

c) Monotonicity: If  $x \leq y$ , then  $N_n(x, z) \leq N_n(y, z)$ . (3.6.4)

d) Associativity:  $N_n(N_n(x, y), z) = N_n(x, N_n(y, z))$ . (3.6.5)

There are cases when not all these axioms are satisfied, for example the associativity when dealing with the neutrosophic normalization after each neutrosophic operation. But, since we work with approximations, we can call these **N-pseudo-norms**, which still give good results in practice.

$N_n$  represent the *and* operator in neutrosophic logic, and respectively the *intersection* operator in neutrosophic set theory.

Let  $J \in \{T, I, F\}$  be a component.

Most known N-norms, as in fuzzy logic and set the T-norms, are:

- The Algebraic Product N-norm:  $N_{n\text{-algebraic}J}(x, y) = x \cdot y$  (3.6.6)

- The Bounded N-Norm:  $N_{n\text{-bounded}J}(x, y) = \max\{0, x + y - 1\}$  (3.6.7)

- The Default (min) N-norm:  $N_{n\text{-min}J}(x, y) = \min\{x, y\}$ . (3.6.8)

A general example of N-norm would be this.

Let  $x(T_1, I_1, F_1)$  and  $y(T_2, I_2, F_2)$  be in the neutrosophic set/logic  $M$ . Then:

$$N_n(x, y) = (T_1 \wedge T_2, I_1 \vee I_2, F_1 \vee F_2) \quad (3.6.9)$$

where the “ $\wedge$ ” operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the “ $\vee$ ” operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the below N-conorms axioms).

For example,  $\wedge$  can be the Algebraic Product T-norm/N-norm, so  $T_1 \wedge T_2 = T_1 \cdot T_2$  (herein we have a product of two subunitary sets – using simplified notation); and  $\vee$  can be the Algebraic Product T-conorm/N-conorm, so  $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$  (herein we have a sum, then a product, and afterwards a subtraction of two subunitary sets).

Or  $\wedge$  can be any T-norm/N-norm, and  $\vee$  any T-conorm/N-conorm from the above and below; for example the easiest way would be to consider the *min* for crisp components (or *inf* for subset components) and respectively *max* for crisp components (or *sup* for subset components).

If we have crisp numbers, we can at the end neutrosophically normalize.

### 3.7. N-conorms

$$N_c: (]0,1^+[\times ]0,1^+[\times ]0,1^+[\rightarrow ]0,1^+[\times ]0,1^+[\times ]0,1^+[$$

$$N_c(x(T_1, I_1, F_1), y(T_2, I_2, F_2)) = (N_cT(x, y), N_cI(x, y), N_cF(x, y)),$$

where  $N_nT(.,.)$ ,  $N_nI(.,.)$ ,  $N_nF(.,.)$  are the truth/membership, indeterminacy, and respectively falsehood/nonmembership components. (3.7.1)

$N_c$  have to satisfy, for any  $x, y, z$  in the neutrosophic logic/set  $M$  of universe of discourse  $U$ , the following axioms:

a) Boundary Conditions:  $N_c(x, \mathbf{1}) = \mathbf{1}$ ,  $N_c(x, \mathbf{0}) = x$ . (3.7.2)

b) Commutativity:  $N_c(x, y) = N_c(y, x)$ . (3.7.3)

c) Monotonicity: if  $x \leq y$ , then  $N_c(x, z) \leq N_c(y, z)$ . (3.7.4)

d) Associativity:  $N_c(N_c(x, y), z) = N_c(x, N_c(y, z))$ . (3.7.5)

There are cases when not all these axioms are satisfied, for example the associativity when dealing with the neutrosophic normalization after each neutrosophic operation. But, since we work with approximations, we can call these **N-pseudo-conorms**, which still give good results in practice.

$N_c$  represent the *or* operator in neutrosophic logic, and respectively the *union* operator in neutrosophic set theory.

Let  $J \in \{T, I, F\}$  be a component.

Most known N-conorms, as in fuzzy logic and set the T-conorms, are:

- The Algebraic Product N-conorm:  $N_{c-\text{algebraic}}J(x, y) = x + y - x \cdot y$  (3.7.6)

- The Bounded N-conorm:  $N_{c-\text{bounded}}J(x, y) = \min\{1, x + y\}$  (3.7.7)

- The Default (max) N-conorm:  $N_{c-\text{max}}J(x, y) = \max\{x, y\}$ . (3.7.8)

A general example of N-conorm would be this.

Let  $x(T_1, I_1, F_1)$  and  $y(T_2, I_2, F_2)$  be in the neutrosophic set/logic  $M$ . Then:

$$N_n(x, y) = (T_1 \vee T_2, I_1 \wedge I_2, F_1 \wedge F_2) \quad (3.7.9)$$

Where – as above - the “ $\wedge$ ” operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the “ $\vee$ ” operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the above N-conorms axioms).

For example,  $\wedge$  can be the Algebraic Product T-norm/N-norm, so  $T_1 \wedge T_2 = T_1 \cdot T_2$  (herein we have a product of two subunitary sets); and  $\vee$  can be the Algebraic Product T-conorm/N-conorm, so  $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$  (herein we have a sum, then a product, and afterwards a subtraction of two subunitary sets).

Or  $\wedge$  can be any T-norm/N-norm, and  $\vee$  any T-conorm/N-conorm from the above; for example the easiest way would be to consider the *min* for crisp components (or *inf* for subset components) and respectively *max* for crisp components (or *sup* for subset components).

If we have crisp numbers, we can at the end neutrosophically normalize.

Since the min/max (or inf/sup) operators work the best for subunitary set components, let's present their definitions below. They are extensions from subunitary intervals {defined in [3]} to any subunitary sets. Analogously we can do for all neutrosophic operators defined in [3].

Let  $x(T_1, I_1, F_1)$  and  $y(T_2, I_2, F_2)$  be in the neutrosophic set/logic M.

**Neutrosophic Conjunction/Intersection:**

$$\begin{aligned} x \wedge y &= (T_{\wedge}, I_{\wedge}, F_{\wedge}), \\ \text{where } \inf T_{\wedge} &= \min\{\inf T_1, \inf T_2\} \\ \sup T_{\wedge} &= \min\{\sup T_1, \sup T_2\} \\ \inf I_{\wedge} &= \max\{\inf I_1, \inf I_2\} \\ \sup I_{\wedge} &= \max\{\sup I_1, \sup I_2\} \\ \inf F_{\wedge} &= \max\{\inf F_1, \inf F_2\} \\ \sup F_{\wedge} &= \max\{\sup F_1, \sup F_2\} \end{aligned} \quad (3.7.10)$$

**Neutrosophic Disjunction/Union:**

$$\begin{aligned} x \vee y &= (T_{\vee}, I_{\vee}, F_{\vee}), \\ \text{where } \inf T_{\vee} &= \max\{\inf T_1, \inf T_2\} \\ \sup T_{\vee} &= \max\{\sup T_1, \sup T_2\} \\ \inf I_{\vee} &= \min\{\inf I_1, \inf I_2\} \\ \sup I_{\vee} &= \min\{\sup I_1, \sup I_2\} \\ \inf F_{\vee} &= \min\{\inf F_1, \inf F_2\} \\ \sup F_{\vee} &= \min\{\sup F_1, \sup F_2\} \end{aligned} \quad (3.7.11)$$

**Neutrosophic Negation/Complement:**

$$\begin{aligned} C(x) &= (T_C, I_C, F_C), \\ \text{where } T_C &= F_1 \\ \inf I_C &= 1 - \sup I_1 \\ \sup I_C &= 1 - \inf I_1 \\ F_C &= T_1 \end{aligned} \quad (3.7.12)$$

Upon the above Neutrosophic Conjunction/Intersection, we can define the

**Neutrosophic Containment:**

We say that the neutrosophic set A is included in the neutrosophic set B of the universe of discourse U,

iff for any  $x(T_A, I_A, F_A) \in A$  with  $x(T_B, I_B, F_B) \in B$  we have:

$$\begin{aligned} \inf T_A &\leq \inf T_B ; \sup T_A \leq \sup T_B; \\ \inf I_A &\geq \inf I_B ; \sup I_A \geq \sup I_B; \\ \inf F_A &\geq \inf F_B ; \sup F_A \geq \sup F_B. \end{aligned} \quad (3.7.13)$$

**3.8. Remarks.**

- a) The non-standard unit interval  $]0, 1^+[$  is merely used for philosophical applications, especially when we want to make a distinction between relative truth (truth in at least one world) and absolute truth (truth in all possible worlds), and similarly for distinction between relative or absolute falsehood, and between relative or absolute indeterminacy.

But, for technical applications of neutrosophic logic and set, the domain of definition and range of the N-norm and N-conorm can be restrained to the normal standard real unit interval  $[0, 1]$ , which is easier to use, therefore:

and

$$N_n: ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$N_c: ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1].$$

- b) Since in NL and NS the sum of the components (in the case when T, I, F are crisp numbers, not sets) is not necessary equal to 1 (so the normalization is not required), we can keep the final result un-normalized.

But, if the normalization is needed for special applications, we can normalize at the end by dividing each component by the sum all components.

If we work with intuitionistic logic/set (when the information is incomplete, i.e. the sum of the crisp components is less than 1, i.e. *sub-normalized*), or with paraconsistent logic/set (when the information overlaps and it is contradictory, i.e. the sum of crisp components is greater than 1, i.e. *over-normalized*), we need to define the neutrosophic measure of a proposition/set.

If  $x(T,I,F)$  is a NL/NS, and T,I,F are crisp numbers in  $[0,1]$ , then the **neutrosophic vector norm** of variable/set x is the sum of its components:

$$N_{\text{vector-norm}}(x) = T+I+F.$$

Now, if we apply the  $N_n$  and  $N_c$  to two propositions/sets which maybe intuitionistic or paraconsistent or normalized (i.e. the sum of components less than 1, bigger than 1, or equal to 1), x and y, what should be the neutrosophic measure of the results  $N_n(x,y)$  and  $N_c(x,y)$  ?

Herein again we have more possibilities:

- either the product of neutrosophic measures of x and y:

$$N_{\text{vector-norm}}(N_n(x,y)) = N_{\text{vector-norm}}(x) \cdot N_{\text{vector-norm}}(y), \quad (3.8.1)$$

- or their average:

$$N_{\text{vector-norm}}(N_n(x,y)) = (N_{\text{vector-norm}}(x) + N_{\text{vector-norm}}(y))/2, \quad (3.8.2)$$

- or other function of the initial neutrosophic measures:

$$N_{\text{vector-norm}}(N_n(x,y)) = f(N_{\text{vector-norm}}(x), N_{\text{vector-norm}}(y)), \quad (3.8.3)$$

where  $f(.,.)$  is a function to be determined according to each application.

Similarly for  $N_{\text{vector-norm}}(N_c(x,y))$ .

Depending on the adopted neutrosophic vector norm, after applying each neutrosophic operator the result is neutrosophically normalized. We'd like to mention that "**neutrosophically normalizing**" doesn't mean that the sum of the resulting crisp components should be 1 as in fuzzy logic/set or intuitionistic fuzzy logic/set, but the sum of the components should be as above: either equal to the product of neutrosophic vector norms of the initial propositions/sets, or equal to the neutrosophic average of the initial propositions/sets vector norms, etc.

In conclusion, we neutrosophically normalize the resulting crisp components  $T', I', F'$  by multiplying each neutrosophic component  $T', I', F'$  with  $S/(T'+I'+F')$ , where

$S = N_{\text{vector-norm}}(N_n(x,y))$  for a N-norm or  $S = N_{\text{vector-norm}}(N_c(x,y))$  for a N-conorm - as defined above.

- c) If T, I, F are subsets of  $[0, 1]$  the problem of neutrosophic normalization is more difficult.
- i) If  $\sup(T)+\sup(I)+\sup(F) < 1$ , we have an *intuitionistic proposition/set*.
  - ii) If  $\inf(T)+\inf(I)+\inf(F) > 1$ , we have a *paraconsistent proposition/set*.
  - iii) If there exist the crisp numbers  $t \in T$ ,  $i \in I$ , and  $f \in F$  such that  $t+i+f=1$ , then we can say that we have a *plausible normalized proposition/set*.

But in many such cases, besides the normalized particular case showed herein, we also have crisp numbers, say  $t_1 \in T$ ,  $i_1 \in I$ , and  $f_1 \in F$  such that  $t_1+i_1+f_1 < 1$  (incomplete information) and  $t_2 \in T$ ,  $i_2 \in I$ , and  $f_2 \in F$  such that  $t_2+i_2+f_2 > 1$  (paraconsistent information).

### 3.9. Examples of Neutrosophic Operators which are N-norms or N-pseudonorms or, respectively N-conorms or N-pseudoconorms.

We define a binary **neutrosophic conjunction (intersection)** operator, which is a particular case of a N-norm (neutrosophic norm, a generalization of the fuzzy T-norm):

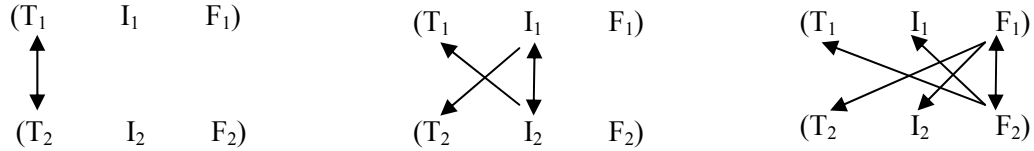
$$c_N^{TIF} : ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$c_N^{TIF}(x, y) = (T_1T_2, I_1I_2 + I_1T_2 + T_1I_2, F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1). \quad (3.9.1)$$

The neutrosophic conjunction (intersection) operator  $x \wedge_N y$  component truth, indeterminacy, and falsehood values result from the multiplication

$$(T_1 + I_1 + F_1) \cdot (T_2 + I_2 + F_2) \quad (3.9.2)$$

since we consider in a prudent way  $T \prec I \prec F$ , where “ $\prec$ ” is a **neutrosophic relationship** and means “weaker”, i.e. the products  $T_iI_j$  will go to  $I$ ,  $T_iF_j$  will go to  $F$ , and  $I_iF_j$  will go to  $F$  for all  $i, j \in \{1,2\}$ ,  $i \neq j$ , while of course the product  $T_1T_2$  will go to  $T$ ,  $I_1I_2$  will go to  $I$ , and  $F_1F_2$  will go to  $F$  (or reciprocally we can say that  $F$  prevails in front of  $I$  which prevails in front of  $T$ , and this neutrosophic relationship is transitive):



Open Table 3.9.1.

So, the truth value is  $T_1T_2$ , the indeterminacy value is  $I_1I_2 + I_1T_2 + T_1I_2$  and the false value is  $F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1$ . The norm of  $x \wedge_N y$  is  $(T_1 + I_1 + F_1) \cdot (T_2 + I_2 + F_2)$ . Thus, if  $x$  and  $y$  are normalized, then  $x \wedge_N y$  is also normalized. Of course, the reader can redefine the neutrosophic conjunction operator, depending on application, in a different way, for example in a more optimistic way, i.e.  $I \prec T \prec F$  or  $T$  prevails with respect to  $I$ , then we get:

$$c_N^{ITF}(x, y) = (T_1T_2 + T_1I_2 + T_2I_1, I_1I_2, F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1). \quad (3.9.3)$$

Or, the reader can consider the order  $T \prec F \prec I$ , etc.

Let's also define the unary **neutrosophic negation** operator:

$$n_N : [0,1] \times [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$n_N(T, I, F) = (F, I, T) \quad (3.9.4)$$

by interchanging the truth  $T$  and falsehood  $F$  vector components.

Similarly, we now define a binary **neutrosophic disjunction (or union)** operator, where we consider the neutrosophic relationship  $F \prec I \prec T$ :

$$d_N^{FIT} : ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$d_N^{FIT}(x, y) = (T_1 T_2 + T_1 I_2 + T_2 I_1 + T_1 F_2 + T_2 F_1, I_1 F_2 + I_2 F_1 + I_1 I_2, F_1 F_2) \quad (3.9.5)$$

We consider as neutrosophic norm of the neutrosophic variable  $x$ , where  $NL(x) = T_1 + I_1 + F_1$ , the sum of its components:  $T_1 + I_1 + F_1$ , which in many cases is 1, but can also be positive  $<1$  or  $>1$ .

Or, the reader can consider the order  $F \prec T \prec I$ , in a pessimistic way, i.e. focusing on indeterminacy  $I$  which prevails in front of the truth  $T$ , or other **neutrosophic order** of the neutrosophic components  $T, I, F$  depending on the application. Therefore,

$$d_N^{FIT}(x, y) = (T_1 T_2 + T_1 F_2 + T_2 F_1, I_1 F_2 + I_2 F_1 + I_1 I_2 + T_1 I_2 + T_2 I_1, F_1 F_2) \quad (3.9.6)$$

### 3.10. Neutrosophic Composition k-Law

Now, we define a more general neutrosophic composition law, named  $k$ -law, in order to be able to define neutrosophic  $k$ -conjunction/intersection and neutrosophic  $k$ -disjunction/union for  $k$  variables, where  $k \geq 2$  is an integer.

Let's consider  $k \geq 2$  neutrosophic variables,  $x_i(T_i, I_i, F_i)$ , for all  $i \in \{1, 2, \dots, k\}$ . Let's denote

$$T = (T_1, \dots, T_k)$$

$$I = (I_1, \dots, I_k)$$

$$F = (F_1, \dots, F_k)$$

We now define a neutrosophic composition law  $o_N$  in the following way:

$$o_N : \{T, I, F\} \rightarrow [0, 1]$$

$$\text{If } z \in \{T, I, F\} \text{ then } z_{o_N} z = \prod_{i=1}^k z_i. \quad (3.10.1)$$

If  $z, w \in \{T, I, F\}$  and  $z \neq w$ , then

$$z_{o_N} w = w_{o_N} z = \sum_{r=1}^{k-1} z_{i_1} \dots z_{i_r} w_{j_{r+1}} \dots w_{j_k} \quad (3.10.2)$$

$$\begin{aligned} & \{i_1, \dots, i_r, j_{r+1}, \dots, j_k\} \equiv \{1, 2, \dots, k\} \\ & (i_1, \dots, i_r) \in C^r(1, 2, \dots, k) \\ & (j_{r+1}, \dots, j_k) \in C^{k-r}(1, 2, \dots, k) \end{aligned}$$

where  $C^r(1, 2, \dots, k)$  means the set of combinations of the elements  $\{1, 2, \dots, k\}$  taken by  $r$ . [Similarly for  $C^{k-r}(1, 2, \dots, k)$ .]

In other words,  $z_{o_N} w$  is the sum of all possible products of the components of vectors  $z$  and  $w$ , such that each product has at least a  $z_i$  factor and at least a  $w_j$  factor, and each product

has exactly  $k$  factors where each factor is a different vector component of  $z$  or of  $w$ . Similarly if we multiply three vectors:

$$T_{o_N} I_{o_N} F = \sum_{\substack{u,v,k-u-v=1 \\ \{i_1, \dots, i_u, j_{u+1}, \dots, j_{u+v}, l_{u+v+1}, \dots, l_k\} \equiv \{1, 2, \dots, k\} \\ (i_1, \dots, i_u) \in C^u(1, 2, \dots, k), (j_{u+1}, \dots, j_{u+v}) \in \\ \in C^v(1, 2, \dots, k), (l_{u+v+1}, \dots, l_k) \in C^{k-u-v}(1, 2, \dots, k)}} T_{i_1 \dots i_u} I_{j_{u+1} \dots j_{u+v}} F_{l_{u+v+1} \dots l_k} \quad (3.10.3)$$

Let's see an example for  $k = 3$ .

$$\begin{aligned} x_1(T_1, I_1, F_1) \\ x_2(T_2, I_2, F_2) \\ x_3(T_3, I_3, F_3) \\ T_{o_N} T = T_1 T_2 T_3, \quad I_{o_N} I = I_1 I_2 I_3, \quad F_{o_N} F = F_1 F_2 F_3 \\ T_{o_N} I = T_1 I_2 I_3 + I_1 T_2 I_3 + I_1 I_2 T_3 + T_1 T_2 I_3 + T_1 I_2 T_3 + I_1 T_2 T_3 \\ T_{o_N} F = T_1 F_2 F_3 + F_1 T_2 F_3 + F_1 F_2 T_3 + T_1 T_2 F_3 + T_1 F_2 T_3 + F_1 T_2 T_3 \\ I_{o_N} F = I_1 F_2 F_3 + F_1 I_2 F_3 + F_1 F_2 I_3 + I_1 I_2 F_3 + I_1 F_2 I_3 + F_1 I_2 I_3 \\ T_{o_N} I_{o_N} F = T_1 I_2 F_3 + T_1 F_2 I_3 + I_1 T_2 F_3 + I_1 F_2 T_3 + F_1 I_2 T_3 + F_1 T_2 I_3 \end{aligned}$$

For the case when indeterminacy  $I$  is not decomposed in subcomponents {as for example  $I = P \cup U$  where  $P$ =paradox (true and false simultaneously) and  $U$ =uncertainty (true or false, not sure which one)}, the previous formulas can be easily written using only three components as:

$$T_{o_N} I_{o_N} F = \sum_{i,j,r \in \mathcal{P}(1,2,3)} T_i I_j F_r \quad (3.10.4)$$

where  $\mathcal{P}(1,2,3)$  means the set of permutations of  $(1,2,3)$  i.e.

$$\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

$$z_{o_N} w = \sum_{\substack{i=1 \\ (i,j,r) \equiv (1,2,3) \\ (j,r) \in \mathcal{P}^2(1,2,3)}}^3 z_i w_j w_{j_r} + w_i z_j z_r \quad (3.10.5)$$

This neurotrophic law is associative and commutative.

### 3.11. Neutrosophic Logic and Set k-Operators

Let's consider the neutrosophic logic crispy values of variables  $x, y, z$  (so, for  $k = 3$ ):

$$NL(x) = (T_1, I_1, F_1) \text{ with } 0 \leq T_1, I_1, F_1 \leq 1 \quad (3.11.1)$$

$$NL(y) = (T_2, I_2, F_2) \text{ with } 0 \leq T_2, I_2, F_2 \leq 1 \quad (3.11.2)$$

$$NL(z) = (T_3, I_3, F_3) \text{ with } 0 \leq T_3, I_3, F_3 \leq 1 \quad (3.11.3)$$

In neutrosophic logic it is not necessary to have the sum of components equals to 1, as in intuitionist fuzzy logic, i.e.  $T_k + I_k + F_k$  is not necessary 1, for  $1 \leq k \leq 3$

As a particular case, we define the tri-nary conjunction neutrosophic operator:

$$c_{3N}^{TIF} : ([0,1] \times [0,1] \times [0,1])^3 \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$c_{3N}^{TIF}(x, y, z) = (T_{o_N} T, I_{o_N} I + I_{o_N} T, F_{o_N} F + F_{o_N} I + F_{o_N} T) \quad (3.11.4)$$

If all  $x, y, z$  are normalized, then  $c_{3N}^{TIF}(x, y, z)$  is also normalized.

If  $x, y$ , or  $z$  are non-normalized, then  $|c_{3N}^{TIF}(x, y, z)| = |x| \cdot |y| \cdot |z|$ , where  $|w|$  means norm of  $w$ .

$c_{3N}^{TIF}$  is a 3-N-norm (neutrosophic norm, i.e. generalization of the fuzzy T-norm).

Again, as a particular case, we define the unary negation neutrosophic operator:

$$n_N : [0,1] \times [0,1] \times [0,1] \rightarrow [0,1] \times [0,1] \times [0,1]$$

$$n_N(x) = n_N(T_1, I_1, F_1) = (F_1, I_1, T_1). \quad (3.11.5)$$

Let's consider the vectors:

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \quad I = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \text{ and } F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}. \quad (3.11.6)$$

$$\text{We note } T_{\bar{x}} = \begin{pmatrix} F_1 \\ T_2 \\ T_3 \end{pmatrix}, \quad T_{\bar{y}} = \begin{pmatrix} T_1 \\ F_2 \\ T_3 \end{pmatrix}, \quad T_{\bar{z}} = \begin{pmatrix} T_1 \\ T_2 \\ F_3 \end{pmatrix}, \quad T_{\bar{xy}} = \begin{pmatrix} F_1 \\ F_2 \\ T_3 \end{pmatrix}, \text{ etc.} \quad (3.11.7)$$

and similarly

$$F_{\bar{x}} = \begin{pmatrix} T_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad F_{\bar{y}} = \begin{pmatrix} F_1 \\ T_2 \\ F_3 \end{pmatrix}, \quad F_{\bar{xz}} = \begin{pmatrix} T_1 \\ F_2 \\ F_3 \end{pmatrix}, \text{ etc.} \quad (3.11.8)$$

For shorter and easier notations let's denote  $z_{o_N} w = zw$  and respectively  $z_{o_N} w_{o_N} v = zwv$  for the vector neutrosophic law defined previously.

Then the neutrosophic tri-nary conjunction/intersection of neutrosophic variables  $x, y$ , and  $z$  is:

$$c_{3N}^{TIF}(x, y, z) = (TT, II + IT, FF + FI + FT + FIT) =$$

$$= (T_1 T_2 T_3, I_1 I_2 I_3 + I_1 I_2 T_3 + I_1 T_2 I_3 + T_1 I_2 I_3 + I_1 T_2 T_3 + T_1 I_2 T_3 + T_1 T_2 I_3,$$

$$F_1 F_2 F_3 + F_1 F_2 I_3 + F_1 I_2 F_3 + I_1 F_2 F_3 + F_1 I_2 I_3 + I_1 F_2 I_3 + I_1 I_2 F_3 +$$

$$+ F_1 F_2 T_3 + F_1 T_2 F_3 + T_1 F_2 F_3 + F_1 T_2 T_3 + T_1 F_2 T_3 + T_1 T_2 F_3 +$$

$$+ T_1 I_2 F_3 + T_1 F_2 I_3 + I_1 F_2 T_3 + I_1 T_2 F_3 + F_1 I_2 T_3 + F_1 T_2 I_3). \quad (3.11.9)$$

Similarly, the neutrosophic tri-nary disjunction/union of neutrosophic variables  $x, y$ , and  $z$  is:

$$d_{3N}^{FIT}(x, y, z) = (TT + TI + TF + TIF, II + IF, FF) =$$



$$\begin{aligned}
& (T_1T_2T_3 + T_1T_2I_3 + T_1I_2T_3 + I_1T_2T_3 + T_1I_2I_3 + I_1T_2I_3 + I_1I_2T_3 + T_1T_2F_3 + T_1F_2T_3 + F_1T_2T_3 + \\
& T_1F_2F_3 + F_1T_2F_3 + F_1F_2T_3 + T_1I_2F_3 + T_1F_2I_3 + I_1F_2T_3 + I_1T_2F_3 + F_1I_2T_3 + F_1T_2I_3, I_1I_2I_3 + I_1I_2F_3 + \\
& I_1F_2I_3 + F_1I_2I_3 + I_1F_2F_3 + F_1I_2F_3 + F_1F_2I_3, F_1F_2F_3)
\end{aligned} \tag{3.11.10}$$

Surely, other neutrosophic orders can be used for tri-nary conjunctions/intersections and respectively for tri-nary disjunctions/unions among the componenets T, I, F.

### 3.12. Neutrosophic Topologies.

A) General Definition of NT:

Let M be a non-empty set.

Let  $x(T_A, I_A, F_A) \in A$  with  $x(T_B, I_B, F_B) \in B$  be in the neutrosophic set/logic M, where A and B are subsets of M. Then (see Section 2.9.1 about N-norms / N-conorms and examples):

$$A \cup B = \{x \in M, x(T_A \vee T_B, I_A \wedge I_B, F_A \wedge F_B)\}, \tag{3.12.1}$$

$$A \cap B = \{x \in M, x(T_A \wedge T_B, I_A \vee I_B, F_A \vee F_B)\}, \tag{3.12.2}$$

$$C(A) = \{x \in M, x(F_A, I_A, T_A)\}. \tag{3.12.3}$$

A General Neutrosophic Topology on the non-empty set M is a family  $\eta$  of Neutrosophic Sets in M satisfying the following axioms:

- $\mathbf{0}(0,0,1)$  and  $\mathbf{1}(1,0,0) \in \eta$ ; (3.12.4)

- If  $A, B \in \eta$ , then  $A \cap B \in \eta$ ; (3.12.5)

- If the family  $\{A_k, k \in K\} \subset \eta$ , then  $\bigcup_{k \in K} A_k \in \eta$ . (3.12.6)

B) An alternative version of NT

-We cal also construct a Neutrosophic Topology on  $NT = ]0, 1^+[$ , considering the associated family of standard or non-standard subsets included in NT, and the empty set  $\emptyset$ , called open sets, which is closed under set union and finite intersection.

Let A, B be two such subsets. The union is defined as:

$A \cup B = A+B-A \cdot B$ , and the intersection as:  $A \cap B = A \cdot B$ . The complement of A,  $C(A) = \{1^+\} - A$ , which is a closed set. {When a non-standard number occurs at an extremity of an internal, one can write “]” instead of “(“ and “[” instead of “)”}. The interval NT, endowed with this topology, forms a *neutrosophic topological space*.

In this example we have used the Algebraic Product N-norm/N-conorm. But other Neutrosophic Topologies can be defined by using various N-norm/N-conorm operators.

In the above defined topologies, if all x's are paraconsistent or respectively intuitionistic, then one has a Neutrosophic Paraconsistent Topology, respectively Neutrosophic Intuitionistic Topology.

### 3.13. Neutrosophic Logic and set used in Image Processing

Neutrosophic logic and set have the advantage of using a third component called “indeterminacy (neutral part)”, which means neither true nor false for a logical proposition, respectively neither membership nor non-membership (but unknown, unsure) of an element with respect with a set. They are generalizations of fuzzy logic and fuzzy set, especially of intuitionistic fuzzy logic and set.

Neutrosophic logic and set have been applied in image processing thanks to their “indeterminacy” neutral component.

#### 3.13.1. Removing Image Noise.

H. D. Cheng and Y. Guo proposed a thresholding algorithm based on neutrosophics that will automatically select the thresholds. The thresholds are needed in order to separate the domains T and F in a neutrosophic value image.

Yanhui Guo, H. D. Cheng, and Yingtao Zhang introduced a Neutrosophic Set filter in order to denoise images. Besides pattern recognition and image vision, denoising an image is highly investigated today.

The image is converted into a neutrosophic set and then one applies a filtering method ( $\gamma$ -median-filtering) in order to reduce the degree of indeterminacy degree of an image; the degree of indeterminacy is found by computing the entropy of the indeterminacy subset. Afterwards the image noise is removed.

A neutrosophic image is composed of pixels, and each pixel P is characterized by three components,  $P_{NS}(T, I, F)$ , where T=degree/percentage of truth, I=degree/percentage of indeterminacy, F=degree/percentage of falsehood.

Then a pixel  $P_{NS}$  situated at the Cartesian coordinates  $(i, j)$  is denoted by  $P_{NS}(i, j)$ . So, we have  $P_{NS}(T(i, j), I(i, j), F(i, j))$ , where  $T(i, j)$  is the probability that pixel  $P_{NS}$  belongs to the white pixel set,  $I(i, j)$  is the probability that pixel  $P_{NS}$  belongs to the indeterminate pixel set, and  $F(i, j)$  is the probability that pixel  $P_{NS}$  belongs to the non-white pixel set. These are defined as follows:

(3.13.1.1-5)

$$T(i, j) = \frac{\bar{g}(i, j) - \bar{g}_{\min}}{\bar{g}_{\max} - \bar{g}_{\min}}$$

$$\bar{g}(i, j) = \frac{1}{w \times w} \sum_{m=i-\frac{w}{2}}^{i+\frac{w}{2}} \sum_{n=j-\frac{w}{2}}^{j+\frac{w}{2}} g(m, n)$$

$$I(i, j) = \frac{\delta(i, j) - \delta_{\min}}{\delta_{\max} - \delta_{\min}}$$

$$\delta(i, j) = \text{abs}(g(i, j) - \bar{g}(i, j))$$

$$F(i, j) = 1 - T(i, j)$$

where  $\bar{g}(i, j)$  is the local mean value of the pixels of the window, while  $\delta(i, j)$  is the absolute value of the difference between intensity  $g(i, j)$  and its local mean value  $\bar{g}(i, j)$ .

*Neutrosophic image entropy* is used – for a gray image - to evaluate the distribution of the gray levels. If the intensity distribution is non-uniform, the entropy is small; but, if the intensities have equal probabilities, the entropy is high.

Neutrosophic image entropy is defined as the sum of the entropies of the three subsets  $T$ ,  $I$ , and  $F$ :

(3.13.1.6-9)

$$En_{NS} = En_T + En_I + En_F$$

$$En_T = - \sum_{i=\min\{T\}}^{\max\{T\}} p_T(i) \ln p_T(i)$$

$$En_I = - \sum_{i=\min\{I\}}^{\max\{I\}} p_I(i) \ln p_I(i)$$

$$En_F = - \sum_{i=\min\{F\}}^{\max\{F\}} p_F(i) \ln p_F(i)$$

where  $En_T$ ,  $En_I$  and  $En_F$  are the entropies of the sets  $T$ ,  $I$ , and  $F$  respectively, while  $p_T(i)$ ,  $p_I(i)$ , and  $p_F(i)$  are the probabilities of elements in  $T$ ,  $I$  and  $F$ , respectively corresponding to  $i$ .

In general the median filter is known for removing the image noise in the gray level domain (Im). The changes in T and F influence and vary the entropy in I, which measures the indeterminacy degree of element  $P_{NS}(i, j)$ .

The result after median filtering,  $\hat{\text{Im}}$ , is defined as:

$$\hat{\text{Im}}(i, j) = \underset{(m,n) \in S_{ij}}{\text{median}} \{ \text{Im}(m, n) \} \quad \text{with} \quad (3.13.1.10)$$

$S_{ij}$  as the neighborhood of the pixel  $(i, j)$ .

Y. Guo et al. proposed the  $\gamma$ -median-filtering operation.

A  $\gamma$ -median-filtering operation for  $P_{NS}$ ,  $\hat{P}_{NS}(\gamma)$ , is defined as:

(3.13.1.11-18)

$$\hat{P}_{NS}(\gamma) = P(\hat{T}(\gamma), \hat{I}(\gamma), \hat{F}(\gamma))$$

$$\hat{T}(\gamma) = \begin{cases} T & I < \gamma \\ \hat{T}_\gamma & I \geq \gamma \end{cases}$$

$$\hat{T}_\gamma(i, j) = \underset{(m,n) \in S_{ij}}{\text{median}}\{T(m, n)\}$$

$$\hat{F}(\gamma) = \begin{cases} F & I < \gamma \\ \hat{F}_\gamma & I \geq \gamma \end{cases}$$

$$\hat{F}_\gamma(i, j) = \underset{(m,n) \in S_{ij}}{\text{median}}\{F(m, n)\}$$

$$\hat{I}_\gamma(i, j) = \frac{\delta_{\hat{T}}(i, j) - \delta_{\hat{T}\min}}{\delta_{\hat{T}\max} - \delta_{\hat{T}\min}}$$

$$\delta_{\hat{T}}(i, j) = \text{abs}\left(\hat{T}(i, j) - \bar{\hat{T}}(i, j)\right)$$

$$\bar{\hat{T}}(i, j) = \frac{1}{w \times w} \sum_{m=i-\frac{w}{2}}^{i+\frac{w}{2}} \sum_{n=j-\frac{w}{2}}^{j+\frac{w}{2}} \hat{T}(m, n)$$

where  $\delta_{\hat{T}}(i, j)$  is the absolute value of the difference between intensity  $\hat{T}(i, j)$  and its local mean value  $\bar{\hat{T}}(i, j)$  at  $(i, j)$  after  $\gamma$ -median-filtering operation.

The new neutrosophic approach to image denoising is described as below:

Step1: Transform the image into  $NS$  domain;

Step2: Use  $\gamma$ -median-filtering operation on the true subset  $T$  to obtain  $\hat{T}_\gamma$ ;

Step 3: Compute the entropy of the indeterminate subset  $\hat{I}_\gamma$ ,  $En_{\hat{I}_\gamma}(i)$ ;

Step 4: If  $\frac{En_{\hat{T}_\gamma}(i+1) - En_{\hat{T}_\gamma}(i)}{En_{\hat{T}_\gamma}(i)} < \delta$ , go to Step 5;

Else  $T = \hat{T}_\gamma$ , go to Step 2;

Step 5: Transform subset  $\hat{T}_\gamma$  from the neutrosophic domain into gray level domain.

The proposed method performs better for removing image noises for those noises whose types are known but also for those noises whose types are unknown.

### 3.13.2. Image Segmentation.

M. Zhang, L. Zhang, and H. D. Cheng have used neutrosophic logic/set in image segmentation. The image description, classification, and recognition depend on the image segmentation – which is used for image analysis/processing, computer vision, and pattern recognition.

Image segmentation means to find objects and boundaries such as curves, lines, etc. and partition a digital image into many regions. By image analysis one locates objects, one measures features, one makes interpretations of scenes.

Image segmentation is done through several methods such as: histogram-based methods, region-based methods, edge-based methods, model-based methods, and watershed methods.

The watershed uses the gradients of an image to split the image into topological areas. It best applies for uniform background and blurred edge objects, whose blurred boundaries are defined in the indeterminacy I.

After removing the noise, the image becomes more uniform.

The image is converted to a neutrosophic set in the following way.

Let  $P_{NS}(i,j)$  be a pixel in the position  $(i,j)$ .

(3.13.1.19)

$T(i,j) =$

$$\begin{aligned}
 & 0, \text{ if } 0 \leq g_{ij} \leq a; \\
 & \frac{(g_{ij} - a)^2}{(b - a)(c - a)}, \text{ if } a \leq g_{ij} \leq b; \\
 & 1 - \frac{(g_{ij} - c)^2}{(c - b)(c - a)}, \text{ if } b \leq g_{ij} \leq c; \\
 & 1, \text{ if } g_{ij} \geq c;
 \end{aligned}$$

and  $F(i,j) = 1 - T(i,j)$ , where  $g_{ij}$  is the intensity value of pixel  $P(i,j)$ .

Then:

- Calculate the histogram of the image.
- Find the local maxima of the histogram.
- Then calculate the mean of local maxima.
- Find the peaks greater than the mean of local maxima.
- Define the low and high limits of the histogram.
- Calculate the parameters  $a$ ,  $b$ ,  $c$  by using the maximum entropy principle: the greater the entropy is the more information the system includes.
- Find two thresholds to separate the domains  $T$  and  $F$ .
- Define homogeneity in intensity domain by using the standard deviation and the discontinuity of the intensity function. Discontinuity measures the changes in gray levels.
- Convert the image to binary image based on  $T$ ,  $I$ ,  $F$ .  
 $T(i,j)$  represents the degree of pixel  $P(i,j)$  to be an object pixel;  
 $I(i,j)$  represents the degree of pixel  $P(i,j)$  to be an edge pixel;  
 $F(i,j)$  represents the degree of pixel  $P(i,j)$  to be a background pixel.  
One determines the sets of object pixels, edge pixels, and background pixels.
- Apply the watershed for converting the binary image in the following way:
  - a) Get the regions  $R_1, R_2, \dots, R_n$  whose pixels are either object pixels, or edge pixels, or background pixels;
  - b) Dilate these regions by using the  $3 \times 3$  structure element;
  - c) At the place where two regions merge, build a dam, until all regions merge together.

Watershed segmentation is good for uniform or nearly uniform images and the edges are connected very well. Yet, watershed method is sensitive to noise and makes over-segmentations.

#### 4) Non-Linear Sequences for Target Tracking

Investigate the possibility of generalizing the result of using Fibonacci sequence (whose terms are linearly recurrent) in linear target tracking to a non-linear recurrent sequence for non-linear tracking (let's say for EKF, UKF, etc.). Then, mathematically studying that non-linear recurrent sequence, we could get improvement of non-linear tracking.

#### Conclusion.

The general idea of the whole paper is to combine, when solving the same military application, many fusion rules and fusion theories, many filtering algorithms and procedures, also fuzzy and neutrosophic logics and sets in order to select the optimum one in each case. In this way we do a unification of fusion rules, theories, modern logics and sets – all of them in order to be able to choosing the best solution to the air force problems.

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## References:

- [1] F. Smarandache and J. Dezert (Eds), *Advances and applications of DSMT for Information Fusion, Vol. 2, Rehoboth, USA*, American Research Press, 2006.
- [2] F. Smarandache and J. Dezert (Eds), *Advances and applications of DSMT for Information Fusion, Vol. 3, Rehoboth, USA*, American Research Press, 2009.
- [3] F. Smarandache F. and J. Dezert, *Information Fusion Based on New Proportional Conflict Redistribution Rules*, Proc. Of Fusion 2005, Philadelphia, July 26-29, 2005.
- [4] A. Tchamova, J. Dezert and F. Smarandache, *A new class of fusion rules based on T-Conorm and T-Norm fuzzy operator*, Information & Security, An International Journal, Vol. 20, 2006.
- [5] L. Zadeh, *On the Validity of Dempster's Rule of Combination of Evidence*, Memo M 79/24, Univ. California, Berkley, 1979.
- [6] M. Florea, *Dempster-Shafer evidence theory through the years: limitations, practical examples, variants under conflict and a new adaptive combination rule*, NATO – Advanced Study Institute “Multisensor Data and Information Processing for Rapid and Robust Situation and Threat Assessment”, Albena-Bulgaria, May 2005.
- [7] F. Smarandache, *Unification of Fusion Theories*, NATO – Advanced Study Institute “Multisensor Data and Information Processing for Rapid and Robust Situation and Threat Assessment”, Albena-Bulgaria, May 2005.
- [8] D. Dubois, H. Prade, *On the combination of evidence in various mathematical frameworks*, Reliability Data Collection and Analysis, J. Flamm and T. Luisi, Brussels, ECSC, EEC, EAFC: pp. 213-241, 1992.
- [9] J. W. Guan, D. A. Bell, *Generalizing the Dempster-Shafer Rule of Combination to Boolean Algebras*, IEEE, 229-236, 1993.
- [10] T. Inagaki, *Interdependence between safety-control policy and multiple-senor schemes via Dempster-Shafer theory*, IEEE Trans. on reliability, Vol. 40, no. 2, pp. 182-188, 1991.
- [11] E. Lefevre, O. Colot, P. Vannoorenberghe, *Belief functions combination and conflict management*, Information Fusion Journal, Elsevier Publisher, Vol. 3, No. 2, pp. 149-162, 2002.
- [12] C. K. Murphy, *Combining belief functions when evidence conflicts*, Decision Support Systems, Elsevier Publisher, Vol. 29, pp. 1-9, 2000.
- [13] K. Sentz, S. Ferson, *Combination of evidence in Dempster-Shafer Theory*, SANDIA Tech. Report, SAND2002-0835, 96 pages, April 2002, [www.sandia.gov/epistemic/Reports/SAND2002-0835.pdf](http://www.sandia.gov/epistemic/Reports/SAND2002-0835.pdf).
- [14] G. Shafer, *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976.
- [15] F. Smarandache, J. Dezert, *Proportional Conflict Redistribution Rule*, arXiv Archives, Los Alamos National Laboratory, July-August 2004, <http://arxiv.org/abs/cs.GL/0408010>, and in PDF format at: [http://arxiv.org/PS\\_cache/cs/pdf/0408/0408010.pdf](http://arxiv.org/PS_cache/cs/pdf/0408/0408010.pdf).
- [16] P. Smets, *Quantified Epistemic Possibility Theory seen as an Hyper Cautious Transferable Belief Model*, <http://iridia.ulb.ac.be/~psmets>.

- [17] R. R. Yager, *Hedging in the combination of evidence*, Journal of Information & Optimization Sciences, Analytic Publishing Co., Vol. 4, No. 1, pp. 73-81, 1983.
- [18] F. Voorbraak, *On the justification of Dempster's rule of combination*, Artificial Intelligence, 48, pp. 171-197, 1991.
- [19] R. R. Yager, *On the relationships of methods of aggregation of evidence in expert systems*, Cybernetics and Systems, Vol. 16, pp. 1-21, 1985.
- [20] L. Zadeh, *Review of Mathematical theory of evidence*, by Glenn Shafer, AI Magazine, Vol. 5, No. 3, pp. 81-83, 1984.
- [21] L. Zadeh, *A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination*, AI Magazine 7, No.2, pp. 85-90, 1986.
- [22] H. D. Cheng and Y. Guo, *A New Neutrosophic Approach to Image Thresholding*, New Mathematics and Natural Computation, Vol. 4, pp. 291-308, 2008.
- [23] Ming Zhang, Ling Zhang, H. D. Cheng, *A Neutrosophic Approach to Image Segmentation Based on Watershed Approach*, Signal Processing, 2009.
- [24] Yanhui Guo, H. D. Cheng, and Yingtao Zhang, *A New Neutrosophic Approach to Image Denoising*, in New Mathematics and Natural Computation, Vol. 5, No. 3, 653-662, 2009.
- [25] F. Smarandache & J. Dezert, *Advances and Applications of DSmt for Information Fusion*, Am. Res. Press, 2004.
- [26] F. Smarandache, *A unifying field in logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*, 1998, 2001, 2003, 2005.
- [27] H. Wang, F. Smarandache, Y.-Q. Zhang, R. Sunderraman, *Interval Neutrosophic Set and Logic: Theory and Applications in Computing*, Hexis, 2005.
- [28] L. Zadeh, *Fuzzy Sets*, Information and Control, Vol. 8, 338-353, 1965.
- [29] Maria Scalzo, Gregory Horvath, Eric Jones, Adnan Bubalo, Mark Alford, Ruixin Niu, Pramod K. Varshney, *Adaptive Filtering for Single Target Tracking*, Signal Processing, Sensor Fusion, and Target Recognition XVIII, edited by Ivan Kadar, Proc. Of SPIE, Vol. 7336, 73360C, 2009.
- [30] Florentin Smarandache, Mark Alford, *A Class of DSmt Conditioning Rules*, Proceedings of COGIS 2009 International Conference, Paris, France, 16-18 November 2009; also presented to this conference; and in arXiv.org: its abstract at <http://xxx.lanl.gov/abs/0908.0100> and the whole paper in PDF format at <http://xxx.lanl.gov/ftp/arxiv/papers/0908/0908.0100.pdf>.